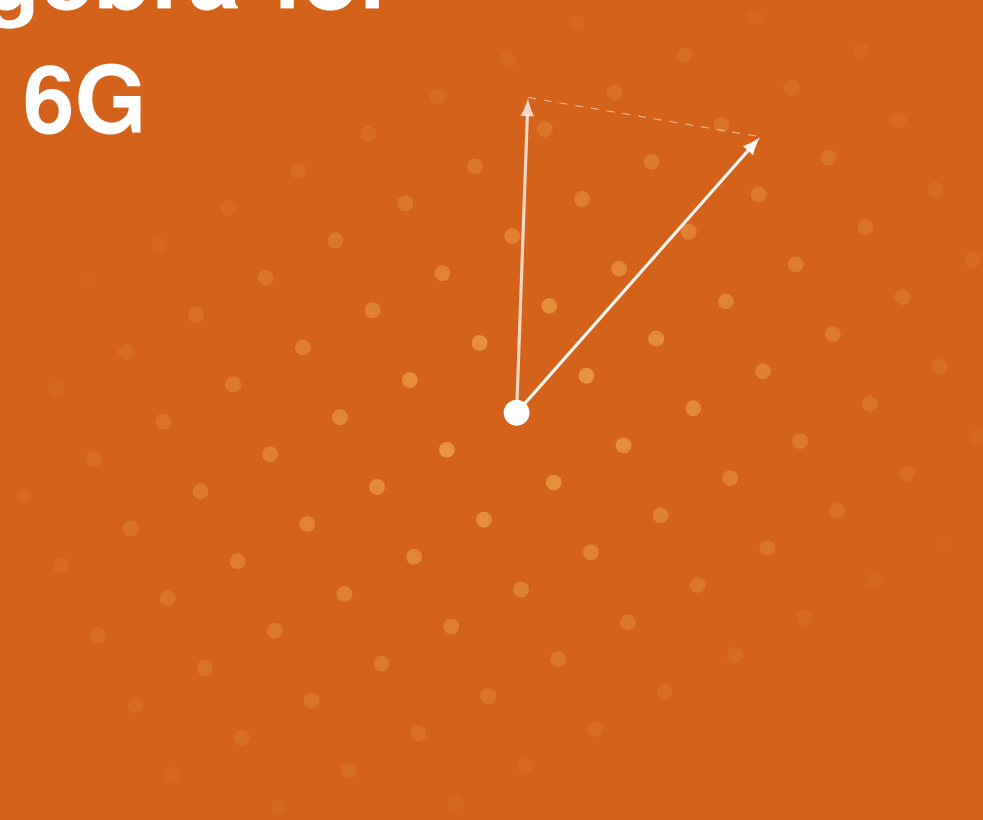
IIT KANPUR

Linear Algebra for AIML and 6G



Foundations, Computation and Applications

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Preface

Linear algebra is the language of modern machine learning and wireless communication. Data samples are vectors, datasets and channels are matrices, and learning and detection are built from matrix products, projections and decompositions.

This book develops linear algebra from first principles and connects each idea to applications in artificial intelligence, machine learning (AIML) and sixth-generation (6G) wireless systems.

How to use this book

- **Definitions** and **theorems** are numbered within each chapter.
- Worked **examples** follow each new idea, with full solutions.
- **Connection** boxes link the mathematics to AIML and 6G.
- Each chapter ends with a **summary** and **exercises**.

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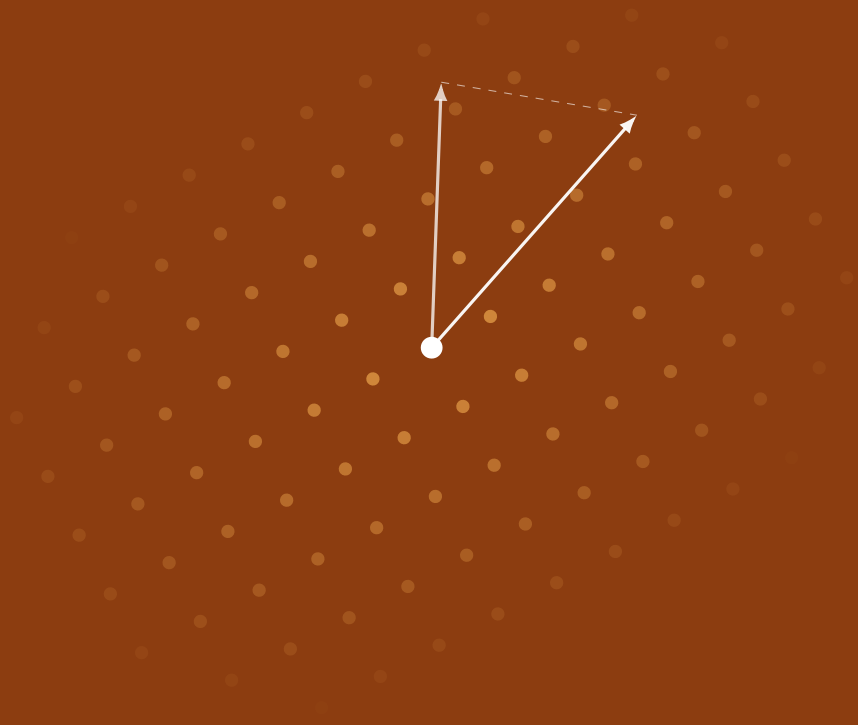
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PART I

Introduction



01

Introduction to Linear Algebra for AIML and 6G

Evolution of AI and ML, the road to 6G, and why linear algebra powers both

In this chapter

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1.3 Linear algebra for AI and ML 7

Two technologies are reshaping the coming decade: artificial intelligence and sixth-generation (6G) wireless networks. Both rest on the same mathematics. A language model and a massive-MIMO base station each spend most of their time multiplying matrices and vectors. This chapter traces how AI and mobile networks evolved, and previews the linear algebra that the rest of the book develops.

1.1 Evolution of AI and ML

Definition 1.1 (AI, ML and deep learning). Four terms describe the field, each narrower than the last:

- **Artificial intelligence** (AI): machines performing tasks that need human intelligence.
- **Machine learning** (ML): learning patterns from data instead of hand-written rules.
- **Deep learning**: machine learning with many-layered neural networks.
- **Generative AI**: models that create text, images, audio and code.

Each is contained in the one before it (Figure 1.1).

1.1.1 Early foundations: 1943–1969

- 1943: McCulloch and Pitts propose a mathematical model of a neuron.
- 1950: Turing asks “Can machines think?”

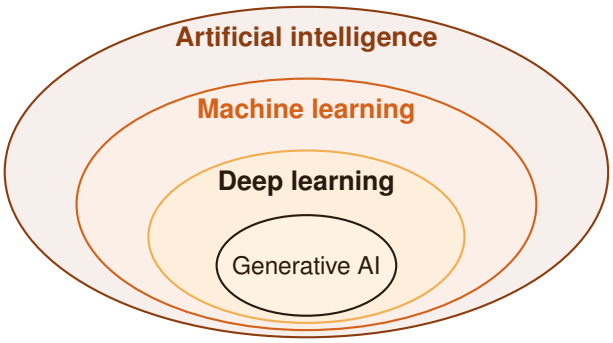


Figure 1.1 Generative AI lies inside deep learning, which lies inside machine learning, which lies inside artificial intelligence.

- 1956: the Dartmouth workshop coins the term *artificial intelligence*.
- 1958: Rosenblatt's **perceptron** learns a linear classifier.
- 1969: Minsky and Papert show the limits of a single perceptron, leading to the first "AI winter".

The perceptron is already pure linear algebra. Given an input vector $\bar{\mathbf{x}}$, a weight vector $\bar{\mathbf{w}}$ and a bias b, it outputs

$$y = \operatorname{sign}\left(\bar{\mathbf{w}}^{\mathsf{T}}\bar{\mathbf{x}} + b\right). \tag{1.1}$$

The set of inputs with $\bar{\mathbf{w}}^{\mathsf{T}}\bar{\mathbf{x}} + b = 0$ is a hyperplane that separates the two classes.

Example 1.1 (A perceptron as a logic gate)

Show that $\bar{\mathbf{w}} = \begin{bmatrix}1 & 1\end{bmatrix}^{\mathsf{T}}$ and $b = -1.5$ implement the AND gate on inputs $x_1, x_2 \in \{0,1\}$.

Solution. The output is $+1$ exactly when $\bar{\mathbf{w}}^{\mathsf{T}}\bar{\mathbf{x}} + b = x_1 + x_2 - 1.5 > 0$. For $\bar{\mathbf{x}} = \begin{bmatrix}1 & 1\end{bmatrix}^{\mathsf{T}}$ this is $0.5 > 0$, so $y = +1$. For $\begin{bmatrix}1 & 0\end{bmatrix}^{\mathsf{T}}$ and $\begin{bmatrix}0 & 1\end{bmatrix}^{\mathsf{T}}$ it is -0.5, and for $\begin{bmatrix}0 & 0\end{bmatrix}^{\mathsf{T}}$ it is -1.5, so $y = -1$. The line $x_1 + x_2 = 1.5$ separates $(1,1)$ from the other three inputs. ◂

> **Remark.** No single line separates the XOR outputs $\{(0,1),(1,0)\}$ from $\{(0,0),(1,1)\}$. This is the limit Minsky and Papert identified; stacking layers removes it.

1.1.2 Statistical learning: 1970s–2000s

- Expert systems encode human knowledge as if–then rules.
- 1986: **backpropagation** makes multilayer neural networks trainable.
- 1989–1998: LeCun's convolutional networks read handwritten digits.
- 1995: **support vector machines** find the maximum-margin hyperplane.
- 2001: **random forests** combine many decision trees.

1.1.3 The deep learning revolution: 2006–2016

Three ingredients came together: big data, GPUs and better training methods.

- 2009: ImageNet provides over 14 million labelled images.
- 2012: AlexNet cuts the ImageNet top-5 error from 26% to 15%.
- 2015: ResNet trains networks with 152 layers.
- 2016: AlphaGo defeats Go champion Lee Sedol.

1.1.4 Foundation models: 2017 onwards

- 2017: the **Transformer** replaces recurrence with attention.
- 2018–2020: BERT and GPT-3 (175 billion parameters) are pre-trained on web-scale text.
- 2022: ChatGPT brings large language models to the public.
- Today: multimodal, reasoning and agentic models are built on the same matrix operations.

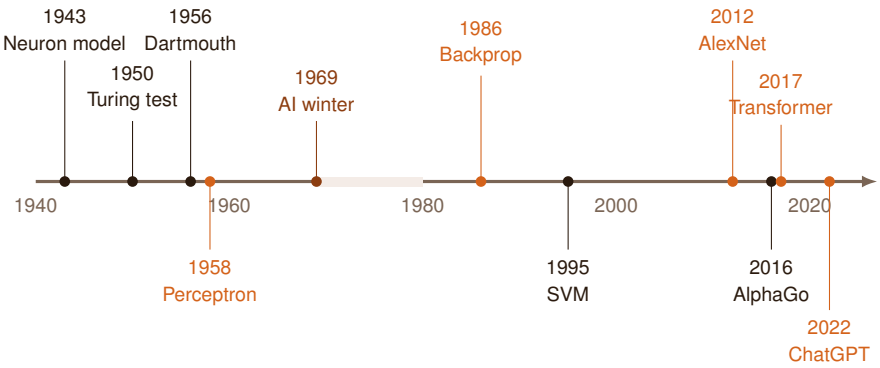


Figure 1.2 Milestones of AI and ML. Each wave was driven by new mathematics, more data and more compute.

1.2 Evolution of 6G

A new generation of mobile networks arrives roughly every decade (Figure 1.3).

1.2.1 1G to 4G

- 1G (1980s): analog voice using FDMA.
- 2G (1991): digital voice and SMS; GSM uses TDMA.
- 3G (2001): mobile internet using WCDMA.
- 4G LTE (2009): all-IP broadband using OFDM and MIMO.

From 4G onwards the physical layer is linear algebra. OFDM is built on the discrete Fourier transform (DFT) matrix, and a multiple-input multiple-output (MIMO) link is the matrix model

$$\bar{\mathbf{y}} = \mathbf{H}\bar{\mathbf{x}} + \bar{\mathbf{n}}, \tag{1.2}$$

where $\bar{\mathbf{x}}$ is the transmitted vector, $\mathbf{H}$ is the channel matrix, $\bar{\mathbf{n}}$ is noise and $\bar{\mathbf{y}}$ is the received vector.

1.2.2 5G: 2019 onwards

The first commercial 5G networks launched in 2019, based on 3GPP Release 15. 5G defines three usage scenarios:

- **eMBB**: enhanced mobile broadband, up to 20 Gbps peak;
- **URLLC**: ultra-reliable low-latency communication, 1 ms latency;
- **mMTC**: massive machine-type communication, 10^6 devices per km^2.

Its key technologies are massive MIMO, mmWave, beamforming and network slicing.

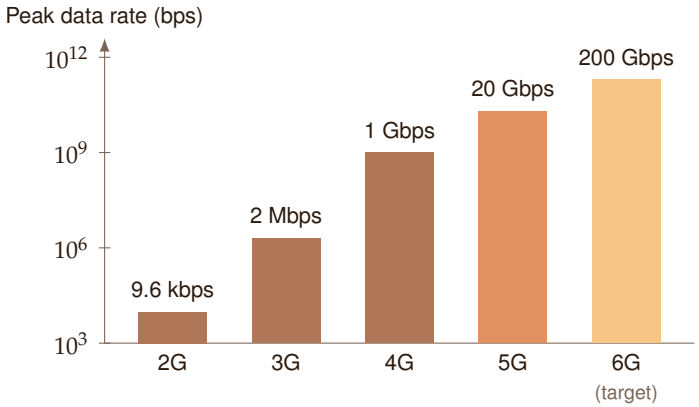


Figure 1.3 Peak data rate of each mobile generation on a logarithmic scale. Rates for 3G–6G are ITU targets.

1.2.3 Vision for 6G: IMT-2030

The ITU-R framework for 6G, called IMT-2030, is set out in Recommendation ITU-R M.2160 (2023). It extends the three 5G scenarios into *immersive communication*, *hyper-reliable and low-latency communication* and *massive communication*, and adds three new ones: *ubiquitous connectivity*, *AI and communication*, and *integrated sensing and communication* (Figure 1.4).

1.2.4 6G targets and timeline

Capability	5G (IMT-2020)	6G (IMT-2030)
Peak data rate	20 Gbps	50–200 Gbps
Latency	1 ms	0.1–1 ms
Connection density	10^6/km^2	10^6–10^8/km^2
Positioning accuracy	—	1–10 cm

Table 1.1 Headline 5G and 6G targets.

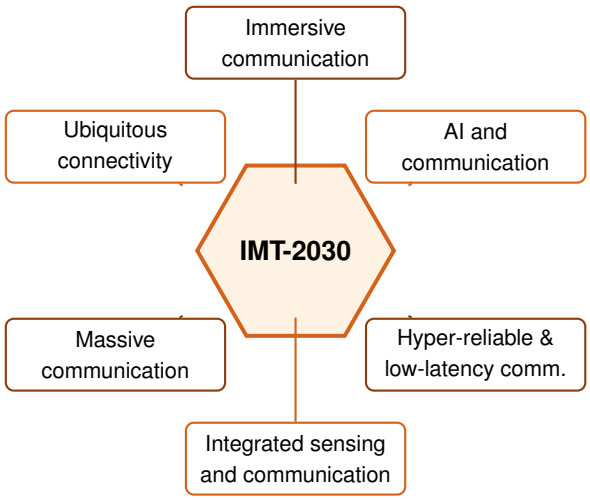


Figure 1.4 The six IMT-2030 usage scenarios. Blue: extended from 5G. Orange: new in 6G.

Standardisation of 6G proceeds in 3GPP:

- Release 20: 6G studies, 2025–2027.
- Release 21: the first 6G specifications. Work items are approved in March 2027, functional freeze is in December 2028, and the final freeze is in March 2029.
- First commercial 6G systems are expected around 2030.

1.2.5 6G enabling technologies

- Sub-THz and THz bands for very wide bandwidths.
- Extremely large and cell-free MIMO, with very large channel matrices.
- Reconfigurable intelligent surfaces that shape the radio channel.
- Integrated sensing and communication (ISAC).
- An AI-native air interface, and non-terrestrial networks (satellites, drones).

AIML & 6G Connection

In a 6G base station with hundreds of antennas, $\mathbf{H}$ in (1.2) has hundreds of rows. Detecting $\bar{\mathbf{x}}$, designing beamformers and estimating $\mathbf{H}$ are all problems in solving linear systems, least squares and eigen-decomposition: the topics of this book.

1.3 Linear algebra for AI and ML

1.3.1 Data as vectors and matrices

- A sample with n features is a vector $\bar{\mathbf{x}} \in \mathbb{R}^n$.
- A dataset of m samples is a matrix $\mathbf{X} \in \mathbb{R}^{m \times n}$, one sample per row.

- A grayscale image is a matrix of pixel values.
- Words and tokens are mapped to **embedding vectors**.

The similarity of two embeddings $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ is measured by the angle between them,

$$\cos\theta = \frac{\bar{\mathbf{u}} \cdot \bar{\mathbf{v}}}{\|\bar{\mathbf{u}}\|\,\|\bar{\mathbf{v}}\|}. \tag{1.3}$$

1.3.2 Linear regression

Linear regression fits a linear model $\bar{\mathbf{y}} \approx \mathbf{X}\bar{\mathbf{w}}$ by minimising the squared error $\|\bar{\mathbf{y}} - \mathbf{X}\bar{\mathbf{w}}\|^2$. The **normal equations** give the least-squares solution

$$\bar{\mathbf{w}} = \left(\mathbf{X}^{\mathsf{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathsf{T}}\bar{\mathbf{y}}. \tag{1.4}$$

This one formula uses the transpose, the inverse, rank and projection.

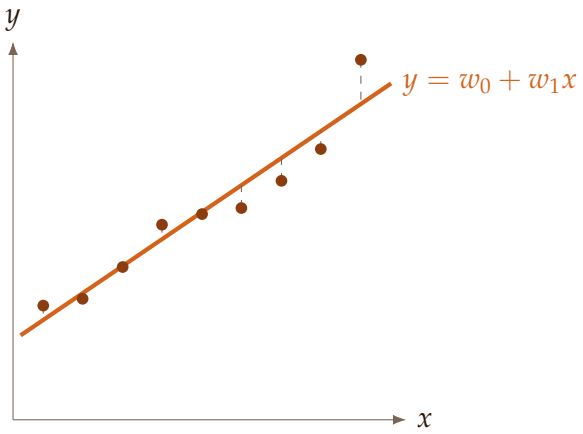


Figure 1.5 The least-squares line minimises the sum of squared vertical residuals.

Example 1.2 (Fitting a line)

Fit $y = w_0 + w_1 x$ to the points $(0,1)$, $(1,3)$ and $(2,4)$.

Solution. Stack the points as

$$\mathbf{X} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \qquad \bar{\mathbf{y}} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \qquad \bar{\mathbf{w}} = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}.$$

Then

$$\mathbf{X}^{\mathsf{T}}\mathbf{X} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}, \qquad \mathbf{X}^{\mathsf{T}}\bar{\mathbf{y}} = \begin{bmatrix} 8 \\ 11 \end{bmatrix}, \qquad \left(\mathbf{X}^{\mathsf{T}}\mathbf{X}\right)^{-1} = \frac{1}{6}\begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix},$$

and (1.4) gives

$$\bar{\mathbf{w}} = \frac{1}{6}\begin{bmatrix} 40 - 33 \\ -24 + 33 \end{bmatrix} = \begin{bmatrix} 7/6 \\ 3/2 \end{bmatrix}.$$

The best line is $y = \frac{7}{6} + \frac{3}{2}x$. Its residuals $-\frac{1}{6}, \frac{1}{3}, -\frac{1}{6}$ sum to zero. ◂

1.3.3 Neural networks

Each layer of a neural network is a matrix–vector product followed by a nonlinearity σ:

$$\bar{\mathbf{h}}_1 = \sigma\left(\mathbf{W}_1 \bar{\mathbf{x}} + \bar{\mathbf{b}}_1\right). \tag{1.5}$$

A deep network composes many such layers (Figure 1.6). Backpropagation computes the gradients as a chain of matrix products, and GPUs are built for fast matrix multiplication.

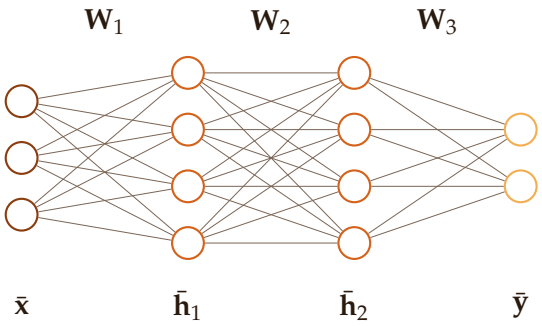


Figure 1.6 A fully connected network. Each set of connections between layers is a weight matrix.

1.3.4 Attention in transformers

In a transformer the token embeddings form the rows of a matrix $\mathbf{X}$. Three weight matrices produce the queries, keys and values,

$$\mathbf{Q} = \mathbf{X}\mathbf{W}_Q, \qquad \mathbf{K} = \mathbf{X}\mathbf{W}_K, \qquad \mathbf{V} = \mathbf{X}\mathbf{W}_V,$$

and attention is

$$\operatorname{Attention}(\mathbf{Q}, \mathbf{K}, \mathbf{V}) = \operatorname{softmax}\left(\frac{\mathbf{Q}\mathbf{K}^{\mathsf{T}}}{\sqrt{d_k}}\right)\mathbf{V}. \tag{1.6}$$

The matrix $\mathbf{Q}\mathbf{K}^{\mathsf{T}}$ holds the inner products of every query with every key. Large language models perform billions of such matrix products.

1.3.5 PCA and the SVD

For centred data $\mathbf{X}$, the covariance matrix and its eigendecomposition are

$$\mathbf{C} = \frac{1}{m}\mathbf{X}^{\mathsf{T}}\mathbf{X} = \mathbf{U}\blacksquare\mathbf{U}^{\mathsf{T}}.$$

The **principal components** are the eigenvectors with the largest eigenvalues (Figure 1.7). The **singular value decomposition**

$$\mathbf{X} = \mathbf{U}\blacksquare\mathbf{V}^{\mathsf{T}} \tag{1.7}$$

gives the best low-rank approximation of $\mathbf{X}$.

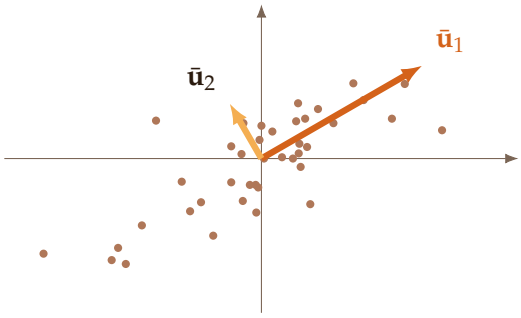


Figure 1.7 Correlated data and its principal directions. $\bar{\mathbf{u}}_1$ captures the most variance; $\bar{\mathbf{u}}_2$ is orthogonal to it.

1.3.6 Linear algebra across this course

Table 1.2 maps the topics of this book to their uses in AI/ML and in 6G.

Topic	Where it is used
Vectors, norms, inner products	Embeddings, similarity, beamforming
Matrices, linear equations	Linear models, MIMO detection
Vector spaces, bases, rank	Features, low-rank models
Eigenvalues, SVD	PCA, recommenders, MIMO capacity
Least squares, pseudo-inverse	Regression, channel estimation
Matrix calculus	Training neural networks

Table 1.2 Linear algebra topics in this book and where they appear in AI/ML and 6G.

Chapter summary

- AI grew from the 1943 neuron model to deep learning and transformers.
- The perceptron $y = \text{sign}(\bar{\mathbf{w}}^{\mathsf{T}}\bar{\mathbf{x}} + b)$ is a linear classifier.
- Mobile networks evolve every decade, from analog 1G to 5G.
- 6G (IMT-2030) adds AI, sensing and ubiquitous coverage; the first specifications freeze in 2029 and systems are expected around 2030.
- MIMO is the matrix model $\bar{\mathbf{y}} = \mathbf{H}\bar{\mathbf{x}} + \bar{\mathbf{n}}$.
- Data, images and embeddings are vectors and matrices.
- Regression, neural networks, attention and PCA are linear algebra at work.

Exercises

1.1 Find a weight vector $\bar{\mathbf{w}}$ and bias b for which the perceptron (1.1) implements the OR gate on $x_1, x_2 \in \{0, 1\}$.

1.2 Explain why no perceptron can implement the XOR gate.

1.3 Fit $y = w_0 + w_1 x$ to the points $(0,2)$, $(1,2)$, $(2,5)$ and $(3,5)$ using the normal equations (1.4).

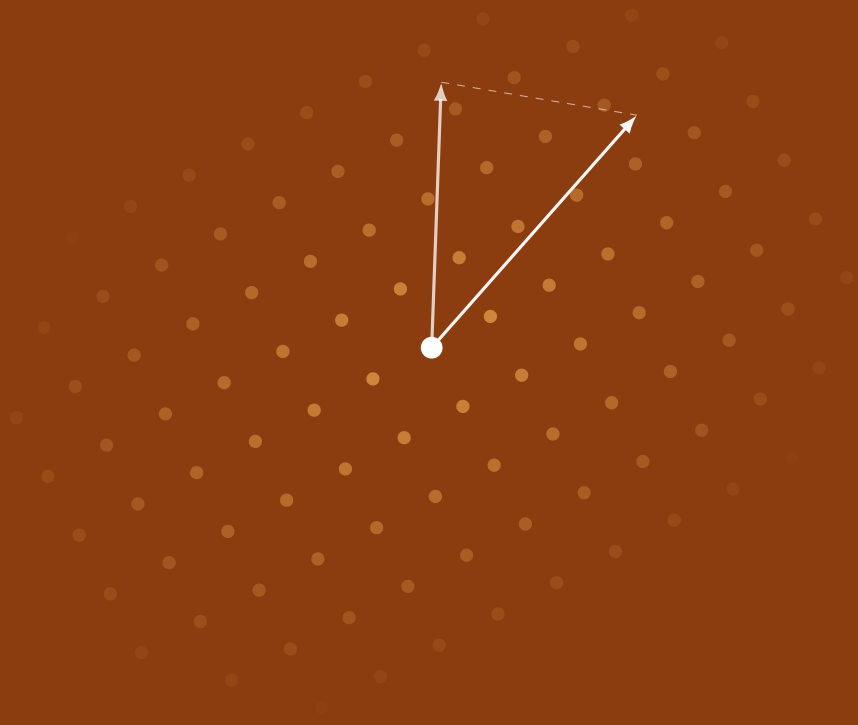
1.4 A layer maps $\bar{\mathbf{x}} \in \mathbb{R}^{784}$ to $\bar{\mathbf{h}}_1 \in \mathbb{R}^{128}$. What is the size of $\mathbf{W}_1$ and how many parameters (weights and biases) does the layer have?

1.5 A sequence of T tokens has embeddings of dimension d, and $\mathbf{W}_Q, \mathbf{W}_K \in \mathbb{R}^{d \times d_k}$. What are the sizes of $\mathbf{Q}$, $\mathbf{K}$ and $\mathbf{Q}\mathbf{K}^{\mathsf{T}}$?

1.6 A MIMO link has 64 base-station antennas and serves 8 single-antenna users. What are the sizes of $\bar{\mathbf{x}}$, $\mathbf{H}$ and $\bar{\mathbf{y}}$ in (1.2) on the uplink?

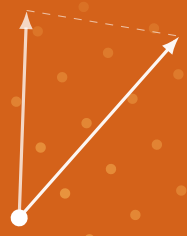
PART II

Vectors and Matrices



02

Vectors



Vectors, operations, dot product, norm, projection and complex vectors

In this chapter

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2.1 Vectors in $\mathbb{R}^n$

A vector collects n numbers into a single object. Vectors describe a data sample in machine learning, a channel between antennas in wireless communication, and a point or a direction in space.

Definition 2.1 (Vector). An n-tuple of real numbers is termed an **n-dimensional vector**. It is denoted as

$$\bar{\mathbf{u}} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^n.$$

The number u_i is the ith **component** of $\bar{\mathbf{u}}$.

Example 2.1 (A vector in $\mathbb{R}^4$)

An example of an $n = 4$ dimensional vector is

$$\bar{\mathbf{u}} = \begin{bmatrix} 5 \\ -3 \\ 2 \\ -6 \end{bmatrix}.$$

Definition 2.2 (Column and row vectors). A **column vector** arranges its components vertically,

$$\bar{\mathbf{u}} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix},$$

while a **row vector** arranges them horizontally,

$$\bar{\mathbf{v}} = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}.$$

By default, all vectors in this book are column vectors.

Definition 2.3 (Transpose). The **transpose** turns a column vector into a row vector, and vice versa:

$$\bar{\mathbf{u}} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \Rightarrow \bar{\mathbf{u}}^{\mathsf{T}} = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix},$$

$$\bar{\mathbf{v}} = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} \Rightarrow \bar{\mathbf{v}}^{\mathsf{T}} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

2.2 Vector operations

Definition 2.4 (Vector sum). The sum of two vectors $\bar{\mathbf{u}}, \bar{\mathbf{v}} \in \mathbb{R}^n$ is formed component by component:

$$\bar{\mathbf{u}} + \bar{\mathbf{v}} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}.$$

Example 2.2 (Vector sum)

Add $\bar{\mathbf{u}} = \begin{bmatrix} 3 & -2 & 6 \end{bmatrix}^\mathsf{T}$ and $\bar{\mathbf{v}} = \begin{bmatrix} 4 & 5 & -8 \end{bmatrix}^\mathsf{T}$.

Solution.

$$\bar{\mathbf{u}} + \bar{\mathbf{v}} = \begin{bmatrix} 3+4 \\ -2+5 \\ 6-8 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ -2 \end{bmatrix}.$$

◀

Definition 2.5 (Scalar multiplication). The product of $\bar{\mathbf{u}} \in \mathbb{R}^n$ with a real number k multiplies every component by k:

$$k\bar{\mathbf{u}} = k \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} ku_1 \\ ku_2 \\ \vdots \\ ku_n \end{bmatrix}.$$

Example 2.3 (Scalar multiplication)

Find $5\bar{\mathbf{u}}$ for $\bar{\mathbf{u}} = \begin{bmatrix} 3 & -2 & 6 \end{bmatrix}^\mathsf{T}$.

Solution.

$$5\bar{\mathbf{u}} = 5 \begin{bmatrix} 3 \\ -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \\ 30 \end{bmatrix}.$$

◀

Definition 2.6 (Linear combination). Given vectors $\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \ldots, \bar{\mathbf{u}}_m \in \mathbb{R}^n$ and real numbers $k_1, k_2, \ldots, k_m$, the vector

$$\bar{\mathbf{v}} = k_1\bar{\mathbf{u}}_1 + k_2\bar{\mathbf{u}}_2 + \cdots + k_m\bar{\mathbf{u}}_m \tag{2.1}$$

is a **linear combination** of $\bar{\mathbf{u}}_1, \ldots, \bar{\mathbf{u}}_m$.

Example 2.4 (Linear combination)

Compute $3\bar{\mathbf{u}} - 2\bar{\mathbf{v}}$ for $\bar{\mathbf{u}} = \begin{bmatrix} 1 & 4 & -3 \end{bmatrix}^\mathsf{T}$ and $\bar{\mathbf{v}} = \begin{bmatrix} 5 & -1 & -2 \end{bmatrix}^\mathsf{T}$.

Solution.

$$3\bar{\mathbf{u}} - 2\bar{\mathbf{v}} = \begin{bmatrix} 3-10 \\ 12+2 \\ -9+4 \end{bmatrix} = \begin{bmatrix} -7 \\ 14 \\ -5 \end{bmatrix}.$$

◀

2.3 Vectors as arrows

In physics, quantities such as force and velocity possess both magnitude and direction. They are represented by arrows of appropriate length and direction. In $\mathbb{R}^3$ the origin $\mathbf{0} = \begin{bmatrix}0 & 0 & 0\end{bmatrix}^\mathsf{T}$ is chosen as the reference point O. Every vector is then uniquely determined by the coordinates of its endpoint (Figure 2.1).

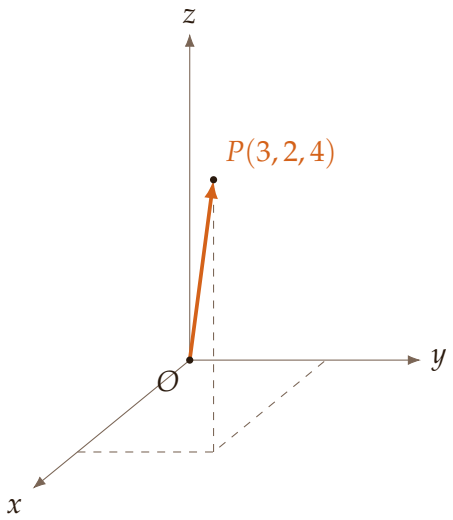


Figure 2.1 The vector $\begin{bmatrix}3 & 2 & 4\end{bmatrix}^\mathsf{T}$ is the arrow from the origin O to the point $P(3,2,4)$.

The resultant $\bar{\mathbf{u}} + \bar{\mathbf{v}}$ of two vectors is obtained by the **parallelogram law**: $\bar{\mathbf{u}} + \bar{\mathbf{v}}$ is the diagonal of the parallelogram formed by $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$. The product $k\bar{\mathbf{u}}$ is obtained by multiplying the length of $\bar{\mathbf{u}}$ by $|k|$, keeping the same direction if $k > 0$ and reversing it if $k < 0$ (Figure 2.2).

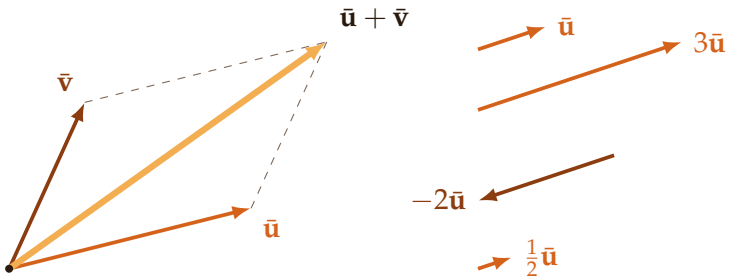


Figure 2.2 Left: the parallelogram law for $\bar{\mathbf{u}} + \bar{\mathbf{v}}$. Right: scaling $\bar{\mathbf{u}}$ by 3, -2 and $\frac{1}{2}$. A negative scalar reverses the direction.

2.4 Properties of vector operations

Theorem 2.7 (Rules of vector arithmetic)

For all $\bar{\mathbf{u}}, \bar{\mathbf{v}}, \bar{\mathbf{w}} \in \mathbb{R}^n$ and real numbers k, k':

1. $(\bar{\mathbf{u}} + \bar{\mathbf{v}}) + \bar{\mathbf{w}} = \bar{\mathbf{u}} + (\bar{\mathbf{v}} + \bar{\mathbf{w}})$ *(associativity)*
2. $\bar{\mathbf{u}} + \mathbf{0} = \mathbf{0} + \bar{\mathbf{u}} = \bar{\mathbf{u}}$ *(zero vector)*

3. $\bar{\mathbf{u}} + (-\bar{\mathbf{u}}) = \mathbf{0}$ *(additive inverse)*
4. $\bar{\mathbf{u}} + \bar{\mathbf{v}} = \bar{\mathbf{v}} + \bar{\mathbf{u}}$ *(commutativity)*
5. $k(\bar{\mathbf{u}} + \bar{\mathbf{v}}) = k\bar{\mathbf{u}} + k\bar{\mathbf{v}}$ *(distributivity over vectors)*
6. $(k + k')\bar{\mathbf{u}} = k\bar{\mathbf{u}} + k'\bar{\mathbf{u}}$ *(distributivity over scalars)*
7. $(kk')\bar{\mathbf{u}} = k(k'\bar{\mathbf{u}})$
8. $1\bar{\mathbf{u}} = \bar{\mathbf{u}}$

Proof. Each rule holds component by component because it holds for real numbers. For example, the ith component of $k(\bar{\mathbf{u}} + \bar{\mathbf{v}})$ is $k(u_i + v_i) = ku_i + kv_i$, the ith component of $k\bar{\mathbf{u}} + k\bar{\mathbf{v}}$. ■

2.5 Dot product and orthogonality

Definition 2.8 (Dot product). The **dot product** (or **inner product**) of $\bar{\mathbf{u}}, \bar{\mathbf{v}} \in \mathbb{R}^n$ is

$$\bar{\mathbf{u}} \cdot \bar{\mathbf{v}} = u_1v_1 + u_2v_2 + \cdots + u_nv_n = \sum_{i=1}^{n} u_iv_i. \tag{2.2}$$

Writing $\bar{\mathbf{u}}^\mathsf{T}$ as a row vector gives the equivalent form

$$\bar{\mathbf{u}}^\mathsf{T}\bar{\mathbf{v}} = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

Example 2.5 (Dot product)

Find $\bar{\mathbf{u}} \cdot \bar{\mathbf{v}}$ for the vectors of Example 2.4.

Solution.

$$\bar{\mathbf{u}} \cdot \bar{\mathbf{v}} = 1 \times 5 + 4 \times (-1) + (-3) \times (-2) = 5 - 4 + 6 = 7.$$

◂

Definition 2.9 (Orthogonality). Vectors $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ are **orthogonal** (perpendicular), written $\bar{\mathbf{u}} \perp \bar{\mathbf{v}}$, if

$$\bar{\mathbf{u}} \cdot \bar{\mathbf{v}} = u_1v_1 + u_2v_2 + \cdots + u_nv_n = 0.$$

Example 2.6 (Orthogonality)

Find k such that $\bar{\mathbf{u}} = \begin{bmatrix} 2 & 3 & 1 & 4 \end{bmatrix}^\mathsf{T}$ and $\bar{\mathbf{v}} = \begin{bmatrix} 5 & k & -7 & -3 \end{bmatrix}^\mathsf{T}$ are orthogonal.

Solution.

$$\bar{\mathbf{u}} \cdot \bar{\mathbf{v}} = 10 + 3k - 7 - 12 = 0 \;\Rightarrow\; 3k = 9 \;\Rightarrow\; k = 3.$$

◂

Theorem 2.10 (Properties of the dot product)

For all $\bar{\mathbf{u}}, \bar{\mathbf{v}}, \bar{\mathbf{w}} \in \mathbb{R}^n$ *and real* k*:*

1. $(\bar{\mathbf{u}} + \bar{\mathbf{v}}) \cdot \bar{\mathbf{w}} = \bar{\mathbf{u}} \cdot \bar{\mathbf{w}} + \bar{\mathbf{v}} \cdot \bar{\mathbf{w}}$
2. $\bar{\mathbf{u}} \cdot \bar{\mathbf{v}} = \bar{\mathbf{v}} \cdot \bar{\mathbf{u}}$
3. $(k\bar{\mathbf{u}}) \cdot \bar{\mathbf{v}} = k(\bar{\mathbf{u}} \cdot \bar{\mathbf{v}})$

2.6 Norm of a vector

Definition 2.11 (Euclidean norm). The **norm** (length) of $\bar{\mathbf{u}} \in \mathbb{R}^n$ is

$$\|\bar{\mathbf{u}}\| = \sqrt{\bar{\mathbf{u}} \cdot \bar{\mathbf{u}}} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2} \tag{2.3}$$

so that $\|\bar{\mathbf{u}}\|^2 = u_1^2 + u_2^2 + \cdots + u_n^2$.

Example 2.7 (Norm of a vector)

Find the norm of $\bar{\mathbf{u}} = \begin{bmatrix} 2 & -3 & 1 & 4 & -1 \end{bmatrix}^\mathsf{T}$.

Solution.

$$\|\bar{\mathbf{u}}\| = \sqrt{2^2 + (-3)^2 + 1^2 + 4^2 + (-1)^2} = \sqrt{4 + 9 + 1 + 16 + 1} = \sqrt{31},$$

so that $\|\bar{\mathbf{u}}\|^2 = 31$.

◂

Definition 2.12 (Unit vector). A vector with $\|\bar{\mathbf{u}}\| = 1$ is a **unit vector**. For any non-zero $\bar{\mathbf{v}}$, the vector

$$\frac{\bar{\mathbf{v}}}{\|\bar{\mathbf{v}}\|}$$

is the unit vector in the direction of $\bar{\mathbf{v}}$. Forming it is called **normalizing** $\bar{\mathbf{v}}$.

Example 2.8 (Unit vector)

Normalize the vector of Example 2.7.

Solution. Since $\|\bar{\mathbf{u}}\| = \sqrt{31}$,

$$\frac{\bar{\mathbf{u}}}{\|\bar{\mathbf{u}}\|} = \frac{1}{\sqrt{31}} \begin{bmatrix} 2 \\ -3 \\ 1 \\ 4 \\ -1 \end{bmatrix}.$$

◂

Theorem 2.13 (Properties of the norm)

For all $\bar{\mathbf{u}}, \bar{\mathbf{v}}, \bar{\mathbf{w}} \in \mathbb{R}^n$ *and real* k*:*

1. Non-negativity and positive definiteness: $\|\bar{\mathbf{u}}\| \geq 0$, *and* $\|\bar{\mathbf{u}}\| = 0 \iff \bar{\mathbf{u}} = \mathbf{0}$.
2. Homogeneity: $\|k\bar{\mathbf{u}}\| = |k|\,\|\bar{\mathbf{u}}\|$.
3. Triangle inequality (subadditivity): $\|\bar{\mathbf{v}} + \bar{\mathbf{w}}\| \leq \|\bar{\mathbf{v}}\| + \|\bar{\mathbf{w}}\|$.

Theorem 2.14 (Cauchy–Schwarz inequality)

For all $\bar{\mathbf{u}}, \bar{\mathbf{v}} \in \mathbb{R}^n$,

$$|\bar{\mathbf{u}} \cdot \bar{\mathbf{v}}| \leq \|\bar{\mathbf{u}}\|\,\|\bar{\mathbf{v}}\|,$$

with equality if and only if one vector is a scalar multiple of the other.

Proof. If $\bar{\mathbf{v}} = \mathbf{0}$ both sides are zero. Otherwise, for every real t,

$$0 \leq \|\bar{\mathbf{u}} - t\bar{\mathbf{v}}\|^2 = \|\bar{\mathbf{u}}\|^2 - 2t\,\bar{\mathbf{u}} \cdot \bar{\mathbf{v}} + t^2\|\bar{\mathbf{v}}\|^2.$$

A quadratic in t that is never negative has a non-positive discriminant:

$$4(\bar{\mathbf{u}} \cdot \bar{\mathbf{v}})^2 - 4\|\bar{\mathbf{u}}\|^2\|\bar{\mathbf{v}}\|^2 \leq 0.$$

■

Theorem 2.14 is equivalent to

$$-\|\bar{\mathbf{u}}\|\,\|\bar{\mathbf{v}}\| \leq \bar{\mathbf{u}} \cdot \bar{\mathbf{v}} \leq \|\bar{\mathbf{u}}\|\,\|\bar{\mathbf{v}}\| \quad \Longrightarrow \quad -1 \leq \frac{\bar{\mathbf{u}} \cdot \bar{\mathbf{v}}}{\|\bar{\mathbf{u}}\|\,\|\bar{\mathbf{v}}\|} \leq 1.$$

2.7 Angle and projection

Because the ratio above lies in $[-1, 1]$, it defines the angle θ between two non-zero vectors:

$$\cos\theta = \frac{\bar{\mathbf{u}} \cdot \bar{\mathbf{v}}}{\|\bar{\mathbf{u}}\|\,\|\bar{\mathbf{v}}\|} \tag{2.4}$$

If $\bar{\mathbf{u}} \perp \bar{\mathbf{v}}$, then $\bar{\mathbf{u}} \cdot \bar{\mathbf{v}} = 0$, so $\cos\theta = 0$ and $\theta = 90°$.

Example 2.9 (Angle between two vectors)

Find the angle between the vectors $\bar{\mathbf{u}} = \begin{bmatrix} 3 & -2 & 2 \end{bmatrix}^{\mathsf{T}}$ and $\bar{\mathbf{v}} = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix}^{\mathsf{T}}$.

Solution. $\bar{\mathbf{u}} \cdot \bar{\mathbf{v}} = 3 - 4 + 8 = 7$, $\|\bar{\mathbf{u}}\| = \sqrt{17}$ and $\|\bar{\mathbf{v}}\| = \sqrt{21}$. From (2.4),

$$\cos\theta = \frac{7}{\sqrt{17} \times \sqrt{21}} = \frac{7}{\sqrt{357}} \approx 0.370, \qquad \theta \approx 68.3°.$$

◀

AIML & 6G Connection

In language models every word is mapped to an *embedding vector*. The cosine in (2.4), called **cosine similarity**, measures how close two meanings are. In a multi-antenna receiver the same inner product $\bar{\mathbf{w}}^{\mathsf{H}}\bar{\mathbf{h}}$ measures how well a beamformer $\bar{\mathbf{w}}$ matches the channel $\bar{\mathbf{h}}$.

Definition 2.15 (Projection onto a vector). The **projection** of $\bar{\mathbf{u}}$ onto a non-zero vector $\bar{\mathbf{v}}$ is the component of $\bar{\mathbf{u}}$ along the unit vector $\bar{\mathbf{v}}/\|\bar{\mathbf{v}}\|$:

$$\operatorname{proj}_{\bar{\mathbf{v}}}\bar{\mathbf{u}} = \left(\bar{\mathbf{u}}\cdot\frac{\bar{\mathbf{v}}}{\|\bar{\mathbf{v}}\|}\right)\frac{\bar{\mathbf{v}}}{\|\bar{\mathbf{v}}\|} = \frac{\bar{\mathbf{u}}\cdot\bar{\mathbf{v}}}{\|\bar{\mathbf{v}}\|^2}\,\bar{\mathbf{v}}. \tag{2.5}$$

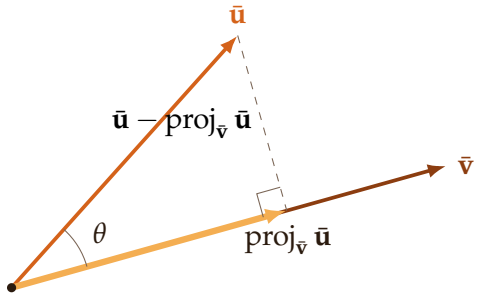


Figure 2.3 Projection of $\bar{\mathbf{u}}$ onto $\bar{\mathbf{v}}$. The residual is orthogonal to $\bar{\mathbf{v}}$.

Example 2.10 (Projection)

Project $\bar{\mathbf{u}}$ onto $\bar{\mathbf{v}}$ for the vectors of Example 2.9.

Solution. With $\bar{\mathbf{u}}\cdot\bar{\mathbf{v}} = 7$ and $\|\bar{\mathbf{v}}\|^2 = 21$, (2.5) gives

$$\operatorname{proj}_{\bar{\mathbf{v}}}\bar{\mathbf{u}} = \left(\frac{1}{\sqrt{21}}\right)^2 \times 7 \times \begin{bmatrix}1\\2\\4\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1\\2\\4\end{bmatrix}.$$

The residual $\bar{\mathbf{u}} - \operatorname{proj}_{\bar{\mathbf{v}}}\bar{\mathbf{u}} = \frac{1}{3}\begin{bmatrix}8 & -8 & 2\end{bmatrix}^{\mathsf{T}}$ satisfies $(8-16+8)/3 = 0$ when dotted with $\bar{\mathbf{v}}$, as Figure 2.3 shows. ◂

Remark. The projection is the point on the line through $\bar{\mathbf{v}}$ closest to $\bar{\mathbf{u}}$. This idea becomes least squares in later chapters.

2.8 Complex vectors

Definition 2.16 (Complex vector). A **complex vector** $\bar{\mathbf{u}} \in \mathbb{C}^n$ has complex numbers as

components,

$$\bar{\mathbf{u}} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \qquad u_i \in \mathbb{C}.$$

Definition 2.17 (Hermitian transpose). The **Hermitian transpose** of $\bar{\mathbf{u}}$ is its conjugate transpose,

$$\bar{\mathbf{u}}^{\mathsf{H}} = \begin{bmatrix} u_1^* & u_2^* & \dots & u_n^* \end{bmatrix}.$$

Example 2.11 (Hermitian transpose)

Find $\bar{\mathbf{u}}^{\mathsf{H}}$ for $\bar{\mathbf{u}} = \begin{bmatrix} 5+i & 2-3i & 1+3i \end{bmatrix}^{\mathsf{T}}$.

Solution. Conjugate each component and transpose:

$$\bar{\mathbf{u}}^{\mathsf{H}} = \begin{bmatrix} 5-i & 2+3i & 1-3i \end{bmatrix}.$$

◀

Definition 2.18 (Complex inner product and norm). The **inner product** of $\bar{\mathbf{u}}, \bar{\mathbf{v}} \in \mathbb{C}^n$ is

$$\bar{\mathbf{u}}^{\mathsf{H}}\bar{\mathbf{v}} = \begin{bmatrix} u_1^* & u_2^* & \dots & u_n^* \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1^* v_1 + u_2^* v_2 + \cdots + u_n^* v_n,$$

and the norm of $\bar{\mathbf{u}}$ is

$$\|\bar{\mathbf{u}}\| = \sqrt{\bar{\mathbf{u}}^{\mathsf{H}}\bar{\mathbf{u}}} = \sqrt{u_1^* u_1 + u_2^* u_2 + \cdots + u_n^* u_n} = \sqrt{|u_1|^2 + |u_2|^2 + \cdots + |u_n|^2}. \quad (2.6)$$

Remark. Without the conjugate, $\bar{\mathbf{u}}^{\mathsf{T}}\bar{\mathbf{u}}$ can be complex or even zero for a non-zero $\bar{\mathbf{u}}$; for example $\begin{bmatrix} 1 & i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = 1 + i^2 = 0$. The conjugate makes $\bar{\mathbf{u}}^{\mathsf{H}}\bar{\mathbf{u}} = \sum_i |u_i|^2$ real and non-negative.

Example 2.12 (Norm of a complex vector)

Find the norm of the vector of Example 2.11.

Solution.

$$\|\bar{\mathbf{u}}\| = \sqrt{5^2 + 1^2 + 2^2 + 3^2 + 1^2 + 3^2} = \sqrt{49} = 7.$$

◀

AIML & 6G Connection

Wireless channels are modelled as complex vectors: each component $h_i = |h_i|e^{j\phi_i}$ carries the gain and phase from one antenna. The received signal power after beamforming is $|\bar{\mathbf{w}}^{\mathsf{H}}\bar{\mathbf{h}}|^2$, and the channel strength is $\|\bar{\mathbf{h}}\|^2 = \sum_i |h_i|^2$.

Chapter summary

- A vector $\bar{\mathbf{u}} \in \mathbb{R}^n$ is an n-tuple of real numbers, a column by default.
- The transpose $\bar{\mathbf{u}}^{\mathsf{T}}$ turns a column vector into a row vector, and vice versa.
- Vectors add component by component; $k\bar{\mathbf{u}}$ multiplies every component by k.
- A linear combination is $k_1\bar{\mathbf{u}}_1 + k_2\bar{\mathbf{u}}_2 + \cdots + k_m\bar{\mathbf{u}}_m$.
- Geometrically, $\bar{\mathbf{u}} + \bar{\mathbf{v}}$ follows the parallelogram law; $k\bar{\mathbf{u}}$ flips direction if $k < 0$.
- Vector addition is commutative and associative; scalars distribute over sums.
- Dot product: $\bar{\mathbf{u}} \cdot \bar{\mathbf{v}} = \bar{\mathbf{u}}^{\mathsf{T}}\bar{\mathbf{v}} = \sum_i u_i v_i$; orthogonal if $\bar{\mathbf{u}} \cdot \bar{\mathbf{v}} = 0$.
- Norm: $\|\bar{\mathbf{u}}\| = \sqrt{\bar{\mathbf{u}} \cdot \bar{\mathbf{u}}}$; it is non-negative, homogeneous and obeys the triangle inequality.
- Cauchy–Schwarz: $|\bar{\mathbf{u}} \cdot \bar{\mathbf{v}}| \leq \|\bar{\mathbf{u}}\|\|\bar{\mathbf{v}}\|$.
- Angle: $\cos\theta = \bar{\mathbf{u}} \cdot \bar{\mathbf{v}} / (\|\bar{\mathbf{u}}\|\|\bar{\mathbf{v}}\|)$; projection: $\text{proj}_{\bar{\mathbf{v}}}\,\bar{\mathbf{u}} = (\bar{\mathbf{u}} \cdot \bar{\mathbf{v}})\bar{\mathbf{v}} / \|\bar{\mathbf{v}}\|^2$.
- Complex vectors: inner product $\bar{\mathbf{u}}^{\mathsf{H}}\bar{\mathbf{v}}$ and norm $\|\bar{\mathbf{u}}\| = \sqrt{\bar{\mathbf{u}}^{\mathsf{H}}\bar{\mathbf{u}}}$.

Exercises

2.1 For $\bar{\mathbf{u}} = \begin{bmatrix} 2 & -1 & 3 \end{bmatrix}^{\mathsf{T}}$ and $\bar{\mathbf{v}} = \begin{bmatrix} 1 & 4 & -2 \end{bmatrix}^{\mathsf{T}}$, compute $2\bar{\mathbf{u}} + 3\bar{\mathbf{v}}$ and $\bar{\mathbf{u}} \cdot \bar{\mathbf{v}}$.

2.2 Find k such that $\begin{bmatrix} 1 & k & -2 & 5 \end{bmatrix}^{\mathsf{T}}$ and $\begin{bmatrix} 4 & 3 & k & -2 \end{bmatrix}^{\mathsf{T}}$ are orthogonal.

2.3 Find the norm of $\begin{bmatrix} 1 & -2 & 2 & 4 \end{bmatrix}^{\mathsf{T}}$ and the unit vector in its direction.

2.4 Find the angle between $\begin{bmatrix} 2 & 0 & -1 \end{bmatrix}^{\mathsf{T}}$ and $\begin{bmatrix} 1 & 3 & 2 \end{bmatrix}^{\mathsf{T}}$.

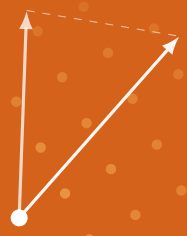
2.5 Project $\begin{bmatrix} 4 & 1 & 3 \end{bmatrix}^{\mathsf{T}}$ onto $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^{\mathsf{T}}$ and verify that the residual is orthogonal to $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^{\mathsf{T}}$.

2.6 For $\bar{\mathbf{u}} = \begin{bmatrix} 1+i & 2-i & 3i \end{bmatrix}^{\mathsf{T}}$, find $\bar{\mathbf{u}}^{\mathsf{H}}$ and $\|\bar{\mathbf{u}}\|$.

2.7 Show that $\|\bar{\mathbf{u}} + \bar{\mathbf{v}}\| \leq \|\bar{\mathbf{u}}\| + \|\bar{\mathbf{v}}\|$ using Theorem 2.14.

03

Matrices



Matrix operations, products, transpose, trace, inverse and special matrices

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3.1 Matrices

Definition 3.1 (Matrix). An $m \times n$ **matrix** $\mathbf{A}$ is a rectangular array of scalars with m rows and n columns,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

Its **size** is $m \times n$. The entry a_{ij} lies in row i and column j.

The **rows** of $\mathbf{A}$ are the $1 \times n$ arrays

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \end{bmatrix}, \begin{bmatrix} a_{21} & a_{22} & \dots & a_{2n} \end{bmatrix}, \dots, \begin{bmatrix} a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix},$$

and its **columns** are the $m \times 1$ arrays

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

Example 3.1 (Rows, columns and entries)

Describe the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & -1 & 3 \\ -2 & 6 & -7 \end{bmatrix}.$$

Solution. $\mathbf{A}$ is 2×3: it has 2 rows and 3 columns. Its rows are $\begin{bmatrix} 4 & -1 & 3 \end{bmatrix}$ and $\begin{bmatrix} -2 & 6 & -7 \end{bmatrix}$, and its columns are

$$\begin{bmatrix} 4 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 6 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ -7 \end{bmatrix}.$$

Two of its entries are $a_{21} = -2$ and $a_{13} = 3$. ◂

Definition 3.2 (Row vector, column vector and zero matrix). A matrix with a single row is a **row vector** of size $1 \times n$, for example $\begin{bmatrix} 3 & -7 & 1 & 5 \end{bmatrix}$. A matrix with a single column is a **column vector** of size $m \times 1$, for example $\bar{\mathbf{u}} = \begin{bmatrix} -4 & 2 & 5 & -1 \end{bmatrix}^{\mathsf{T}}$. A matrix whose entries are all zero is a **zero matrix**, denoted $\mathbf{0}$:

$$\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

3.2 Addition and scalar multiplication

Definition 3.3 (Matrix addition and scalar multiple). Let $\mathbf{A}$ and $\mathbf{B}$ be matrices of the same size $m \times n$ and let k be a scalar. The sum $\mathbf{A} + \mathbf{B}$ and the scalar multiple $k\mathbf{A}$ are formed entry by entry:

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} a_{11} + b_{11} & \ldots & a_{1n} + b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & \ldots & a_{mn} + b_{mn} \end{bmatrix}, \qquad k\mathbf{A} = \begin{bmatrix} ka_{11} & \ldots & ka_{1n} \\ \vdots & \ddots & \vdots \\ ka_{m1} & \ldots & ka_{mn} \end{bmatrix}.$$

The **negative** is $-\mathbf{A} = (-1)\mathbf{A}$, and **subtraction** is $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$, with entries $a_{ij} - b_{ij}$.

Theorem 3.4 (Rules of addition and scalar multiplication)

For matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *of the same size and scalars* k, k'*:*

1. $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$

2. $\mathbf{A}+\mathbf{0}=\mathbf{0}+\mathbf{A}=\mathbf{A}$
3. $\mathbf{A}+(-\mathbf{A})=(-\mathbf{A})+\mathbf{A}=\mathbf{0}$
4. $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$
5. $k(\mathbf{A}+\mathbf{B})=k\mathbf{A}+k\mathbf{B}$
6. $(k+k')\mathbf{A}=k\mathbf{A}+k'\mathbf{A}$
7. $(kk')\mathbf{A}=k(k'\mathbf{A})$
8. $1\mathbf{A}=\mathbf{A}$

3.3 Matrix multiplication

The product $\mathbf{AB}$ is defined only when the number of columns of $\mathbf{A}$ equals the number of rows of $\mathbf{B}$.

Definition 3.5 (Matrix product). If $\mathbf{A}$ is $m\times p$ and $\mathbf{B}$ is $p\times n$, the product $\mathbf{C}=\mathbf{AB}$ is the $m\times n$ matrix whose ij-entry is the ith row of $\mathbf{A}$ times the jth column of $\mathbf{B}$:

$$c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\cdots+a_{ip}b_{pj}=\sum_{k=1}^{p}a_{ik}b_{kj} \tag{3.1}$$

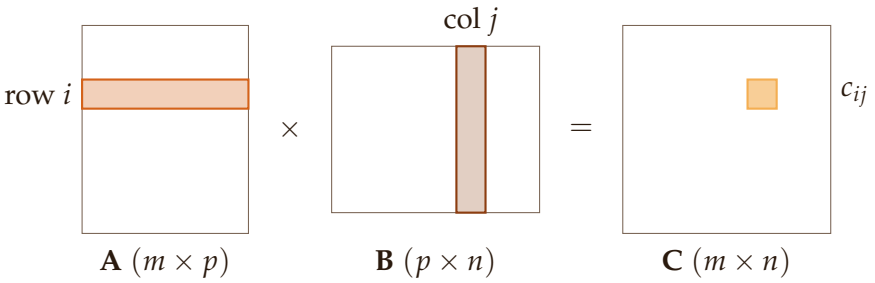


Figure 3.1 The entry c_{ij} of $\mathbf{AB}$ pairs row i of $\mathbf{A}$ with column j of $\mathbf{B}$.

Example 3.2 (Matrix product)

Compute $\mathbf{AB}$ for $\mathbf{A}=\begin{bmatrix}2&1\\-1&3\end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix}3&-2&1\\4&0&-5\end{bmatrix}$.

Solution. $\mathbf{A}$ is 2×2 and $\mathbf{B}$ is 2×3, so the product is defined and is 2×3:

$$\mathbf{AB}=\begin{bmatrix}6+4&-4+0&2-5\\-3+12&2+0&-1-15\end{bmatrix}=\begin{bmatrix}10&-4&-3\\9&2&-16\end{bmatrix}.$$

◀

Example 3.3 (The order of multiplication matters)

Compute $\mathbf{AB}$ and $\mathbf{BA}$ for $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 4 & -1 \\ 2 & 5 \end{bmatrix}$.

Solution.

$$\mathbf{AB} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 10 & 3 \\ 10 & 14 \end{bmatrix}, \qquad \mathbf{BA} = \begin{bmatrix} 4 & -1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 9 & 17 \end{bmatrix}.$$

◀

Example 3.3 shows that, in general, $\mathbf{AB} \neq \mathbf{BA}$: matrix multiplication is *not* commutative.

Theorem 3.6 (Properties of the matrix product)

Whenever the sums and products are defined, and for any scalar k:

1. $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$
2. $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
3. $(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$
4. $k(\mathbf{AB}) = (k\mathbf{A})\mathbf{B} = \mathbf{A}(k\mathbf{B})$
5. $\mathbf{0A} = \mathbf{A0} = \mathbf{0}$

AIML & 6G Connection

A MIMO link with N_t transmit and N_r receive antennas is modelled as $\bar{\mathbf{y}} = \mathbf{H}\bar{\mathbf{x}} + \bar{\mathbf{n}}$, where $\mathbf{H}$ is the $N_r \times N_t$ channel matrix. A neural-network layer computes $\sigma(\mathbf{W}\bar{\mathbf{x}} + \bar{\mathbf{b}})$: the same matrix–vector product.

3.4 The transpose

Definition 3.7 (Transpose). The **transpose** $\mathbf{A}^{\mathsf{T}}$ of an $m \times n$ matrix $\mathbf{A}$ is the $n \times m$ matrix whose rows are the columns of $\mathbf{A}$:

$$\left[\mathbf{A}^{\mathsf{T}}\right]_{ij} = [\mathbf{A}]_{ji}.$$

In particular, the transpose of a row vector is a column vector, and vice versa.

Example 3.4 (Transpose)

$$\begin{bmatrix} 3 & -1 & 2 \\ 0 & 4 & 5 \end{bmatrix}^{\mathsf{T}} = \begin{bmatrix} 3 & 0 \\ -1 & 4 \\ 2 & 5 \end{bmatrix}.$$

Theorem 3.8 (Properties of the transpose)

For matrices of compatible sizes and any scalar k:

1. $(\mathbf{A}+\mathbf{B})^{\mathsf{T}} = \mathbf{A}^{\mathsf{T}} + \mathbf{B}^{\mathsf{T}}$
2. $(k\mathbf{A})^{\mathsf{T}} = k\mathbf{A}^{\mathsf{T}}$
3. $(\mathbf{A}^{\mathsf{T}})^{\mathsf{T}} = \mathbf{A}$
4. $(\mathbf{AB})^{\mathsf{T}} = \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}$

Proof of (4). Let $\mathbf{A}$ be $m \times p$ and $\mathbf{B}$ be $p \times n$. Then

$$\left[(\mathbf{AB})^{\mathsf{T}}\right]_{ij} = [\mathbf{AB}]_{ji} = \sum_{k=1}^{p} a_{jk} b_{ki} = \sum_{k=1}^{p} \left[\mathbf{B}^{\mathsf{T}}\right]_{ik} \left[\mathbf{A}^{\mathsf{T}}\right]_{kj} = \left[\mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\right]_{ij}.$$

■

Operation	Sizes	Result
Addition $\mathbf{A}+\mathbf{B}$	$m \times n$, $m \times n$	$m \times n$
Scalar multiple $k\mathbf{A}$	$m \times n$	$m \times n$
Product $\mathbf{AB}$	$m \times p$, $p \times n$	$m \times n$
Transpose $\mathbf{A}^{\mathsf{T}}$	$m \times n$	$n \times m$
Trace $\operatorname{tr}(\mathbf{A})$	$n \times n$	scalar
Inverse $\mathbf{A}^{-1}$	$n \times n$	$n \times n$

Table 3.1 Sizes of matrix operations.

3.5 Square matrices, trace and the identity

Definition 3.9 (Square matrix, diagonal and trace). A **square matrix** has the same number of rows and columns; its size is $n \times n$. Its **diagonal** entries are $a_{11}, a_{22}, \ldots, a_{nn}$, and its **trace** is their sum,

$$\operatorname{tr}(\mathbf{A}) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}. \tag{3.2}$$

Example 3.5 (Diagonal and trace)

Find the diagonal entries and the trace of $\mathbf{A} = \begin{bmatrix} 3 & 1 & -2 \\ 4 & -5 & 0 \\ 2 & 6 & 1 \end{bmatrix}$.

Solution. The diagonal entries are 3, -5 and 1, so $\operatorname{tr}(\mathbf{A}) = 3 - 5 + 1 = -1$. ◂

Theorem 3.10 (Properties of the trace)

For square matrices $\mathbf{A}, \mathbf{B}$ *of the same size and any scalar* k*:*

1. $\operatorname{tr}(\mathbf{A}+\mathbf{B}) = \operatorname{tr}(\mathbf{A}) + \operatorname{tr}(\mathbf{B})$
2. $\operatorname{tr}(\mathbf{A}^{\mathsf{T}}) = \operatorname{tr}(\mathbf{A})$
3. $\operatorname{tr}(k\mathbf{A}) = k\ \operatorname{tr}(\mathbf{A})$
4. $\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA})$

Definition 3.11 (Identity matrix). The $n \times n$ **identity matrix** $\mathbf{I}_n$ has every diagonal entry equal to 1 and every other entry equal to 0. For example,

$$\mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The identity leaves a matrix unchanged under multiplication. If $\mathbf{A}$ is $n \times n$, then $\mathbf{AI}_n = \mathbf{I}_n\mathbf{A} = \mathbf{A}$; if $\mathbf{A}$ is $m \times n$, then

$$\mathbf{AI}_n = \mathbf{I}_m\mathbf{A} = \mathbf{A}.$$

3.6 The inverse of a matrix

Definition 3.12 (Invertible matrix). A square matrix $\mathbf{A}$ is **invertible** (or **non-singular**) if there exists a matrix $\mathbf{B}$ such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}.$$

Then $\mathbf{B}$ is the **inverse** of $\mathbf{A}$, written $\mathbf{A}^{-1}$.

Theorem 3.13 (Properties of the inverse)

1. *If the inverse of* $\mathbf{A}$ *exists, it is unique.*
2. *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are invertible, then so is* $\mathbf{AB}$*, and* $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Proof. (1) If $\mathbf{B}_1$ and $\mathbf{B}_2$ are both inverses of $\mathbf{A}$, then

$$\mathbf{B}_1 = \mathbf{B}_1(\mathbf{AB}_2) = (\mathbf{B}_1\mathbf{A})\mathbf{B}_2 = \mathbf{B}_2.$$

(2) Using associativity,

$$(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{A}(\mathbf{BB}^{-1})\mathbf{A}^{-1} = \mathbf{AA}^{-1} = \mathbf{I},$$

and similarly $(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) = \mathbf{I}$. ∎

Theorem 3.14 (Inverse of a 2×2 matrix)

If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *and* $ad - bc \neq 0$, *then*

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{3.3}$$

Example 3.6 (Inverse of a 2×2 matrix)

Find the inverse of $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ and verify it.

Solution. Here $ad - bc = 9 - 8 = 1$, so by (3.3)

$$\mathbf{A}^{-1} = \frac{1}{1} \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}.$$

Check:

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

◀

3.7 Diagonal and triangular matrices

Definition 3.15 (Diagonal matrix). A square matrix is **diagonal** if all its off-diagonal entries are zero:

$$\mathbf{D} = \begin{bmatrix} d_{11} & 0 & \dots & 0 \\ 0 & d_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_{nn} \end{bmatrix} = \operatorname{diag}(d_{11}, d_{22}, \dots, d_{nn}).$$

For example, $\begin{bmatrix} -4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \operatorname{diag}(-4, 5, 9).$

Definition 3.16 (Triangular matrices). A square matrix is **upper triangular** if every entry below the diagonal is zero, and **lower triangular** if every entry above the diagonal is zero:

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix} \text{ (upper)}, \qquad \begin{bmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \text{ (lower)}.$$

3.8 Symmetric and orthogonal matrices

Definition 3.17 (Symmetric and skew-symmetric matrices). A square matrix is **symmetric** if $\mathbf{A}^\mathsf{T} = \mathbf{A}$, and **skew-symmetric** if $\mathbf{A}^\mathsf{T} = -\mathbf{A}$.

Example 3.7 (Symmetric and skew-symmetric)

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & -6 \\ 1 & -2 & 3 \\ -6 & 3 & 7 \end{bmatrix} \text{ is symmetric,}$$

$$\mathbf{B} = \begin{bmatrix} 0 & -2 & 6 \\ 2 & 0 & -1 \\ -6 & 1 & 0 \end{bmatrix} \text{ is skew-symmetric.}$$

The diagonal of a skew-symmetric matrix is zero, since $a_{ii} = -a_{ii}$.

Definition 3.18 (Orthogonal matrix). A real square matrix $\mathbf{A}$ is **orthogonal** if $\mathbf{A}^\mathsf{T} = \mathbf{A}^{-1}$, that is,

$$\mathbf{A}^\mathsf{T}\mathbf{A} = \mathbf{A}\mathbf{A}^\mathsf{T} = \mathbf{I}. \tag{3.4}$$

Theorem 3.19 (Columns of an orthogonal matrix)

*Let $\mathbf{A}$ be orthogonal with columns $\bar{\mathbf{a}}_1, \ldots, \bar{\mathbf{a}}_n$. The columns form an **orthonormal set**:*

$$\|\bar{\mathbf{a}}_i\| = 1, \qquad \bar{\mathbf{a}}_i^\mathsf{T}\bar{\mathbf{a}}_j = 0 \text{ for } i \neq j.$$

The rows of $\mathbf{A}$ also form an orthonormal set.

Proof. The ij-entry of $\mathbf{A}^\mathsf{T}\mathbf{A}$ is $\bar{\mathbf{a}}_i^\mathsf{T}\bar{\mathbf{a}}_j$. Setting $\mathbf{A}^\mathsf{T}\mathbf{A} = \mathbf{I}$ gives $\bar{\mathbf{a}}_i^\mathsf{T}\bar{\mathbf{a}}_i = 1$ and $\bar{\mathbf{a}}_i^\mathsf{T}\bar{\mathbf{a}}_j = 0$ for $i \neq j$. Applying the same argument to $\mathbf{A}\mathbf{A}^\mathsf{T} = \mathbf{I}$ gives the result for the rows. ■

Example 3.8 (An orthogonal matrix)

Show that $\mathbf{A} = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$ is orthogonal and that its columns are orthonormal.

Solution.

$$\mathbf{A}\mathbf{A}^\mathsf{T} = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}\frac{1}{5}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The columns are $\bar{\mathbf{a}}_1 = \frac{1}{5}\begin{bmatrix} 3 & 4 \end{bmatrix}^\mathsf{T}$ and $\bar{\mathbf{a}}_2 = \frac{1}{5}\begin{bmatrix} 4 & -3 \end{bmatrix}^\mathsf{T}$, with $\|\bar{\mathbf{a}}_1\| = \|\bar{\mathbf{a}}_2\| = 1$ and $\bar{\mathbf{a}}_1^\mathsf{T}\bar{\mathbf{a}}_2 = \frac{1}{25}(12 - 12) = 0$. ◂

Example 3.9 (Rotation and reflection)

For every angle θ the matrices

$$\mathbf{A} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

are orthogonal. For $\theta = \pi/6$,

$$\mathbf{A} = \frac{1}{2}\begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix}, \qquad \mathbf{B} = \frac{1}{2}\begin{bmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{bmatrix}.$$

Remark. Since $\|\mathbf{A}\bar{\mathbf{x}}\|^2 = \bar{\mathbf{x}}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{A}\bar{\mathbf{x}} = \|\bar{\mathbf{x}}\|^2$, an orthogonal matrix preserves lengths. Geometrically, $\mathbf{A}$ in Example 3.9 is a rotation and $\mathbf{B}$ is a reflection.

3.9 Complex matrices

A **complex matrix** has complex entries, for example

$$\mathbf{A} = \begin{bmatrix} 3-2i & 1+5i & 7i \\ 4+i & 2-6i & 5-3i \end{bmatrix}.$$

Definition 3.20 (Hermitian transpose). The **Hermitian transpose** $\mathbf{A}^{\mathsf{H}}$ is obtained by transposing $\mathbf{A}$ and taking the complex conjugate of each entry:

$$\left[\mathbf{A}^{\mathsf{H}}\right]_{ij} = a_{ji}^{*}.$$

Example 3.10 (Hermitian transpose)

For the matrix $\mathbf{A}$ above,

$$\mathbf{A}^{\mathsf{H}} = \begin{bmatrix} 3+2i & 4-i \\ 1-5i & 2+6i \\ -7i & 5+3i \end{bmatrix}.$$

Definition 3.21 (Hermitian, skew-Hermitian and unitary matrices). A square complex matrix $\mathbf{A}$ is

- **Hermitian** if $\mathbf{A} = \mathbf{A}^{\mathsf{H}}$;
- **skew-Hermitian** if $\mathbf{A} = -\mathbf{A}^{\mathsf{H}}$;
- **unitary** if $\mathbf{A}^{-1} = \mathbf{A}^{\mathsf{H}}$, that is, $\mathbf{A}\mathbf{A}^{\mathsf{H}} = \mathbf{A}^{\mathsf{H}}\mathbf{A} = \mathbf{I}$.

These are the complex counterparts of symmetric, skew-symmetric and orthogonal matrices.

Example 3.11 (Hermitian and skew-Hermitian)

$$\mathbf{A} = \begin{bmatrix} 5 & 3-2i \\ 3+2i & -1 \end{bmatrix} = \mathbf{A}^{\mathsf{H}}$$

is Hermitian. For $\mathbf{B} = \begin{bmatrix} 0 & 2+3i \\ -2+3i & 0 \end{bmatrix}$,

$$\mathbf{B}^{\mathsf{H}} = \begin{bmatrix} 0 & -2-3i \\ 2-3i & 0 \end{bmatrix} = -\mathbf{B},$$

so $\mathbf{B}$ is skew-Hermitian.

Theorem 3.22 (Columns of a unitary matrix)

Let $\mathbf{A}$ *have columns* $\bar{\mathbf{a}}_1, \ldots, \bar{\mathbf{a}}_n$. *Then* $\mathbf{A}$ *is unitary if and only if its columns form an orthonormal set:*

$$\|\bar{\mathbf{a}}_i\| = 1, \qquad \bar{\mathbf{a}}_i^{\mathsf{H}} \bar{\mathbf{a}}_j = 0 \text{ for } i \neq j.$$

Example 3.12 (A unitary matrix)

Show that $\mathbf{A} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$ is unitary and that its columns are orthonormal.

Solution.

$$\mathbf{A}^{\mathsf{H}}\mathbf{A} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The columns are $\bar{\mathbf{a}}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \end{bmatrix}^{\mathsf{T}}$ and $\bar{\mathbf{a}}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 1 \end{bmatrix}^{\mathsf{T}}$. Then

$$\|\bar{\mathbf{a}}_1\|^2 = \tfrac{1}{2}\left(|1|^2 + |i|^2\right) = 1 = \|\bar{\mathbf{a}}_2\|^2, \qquad \bar{\mathbf{a}}_1^{\mathsf{H}} \bar{\mathbf{a}}_2 = \tfrac{1}{2} \begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = \tfrac{1}{2}(i - i) = 0.$$

◀

AIML & 6G Connection

In OFDM, the basis for 4G, 5G and 6G air interfaces, the transmitter applies the DFT matrix $\mathbf{F}$, which is unitary: $\mathbf{F}^{\mathsf{H}}\mathbf{F} = \mathbf{I}$. The receiver undoes it simply with $\mathbf{F}^{\mathsf{H}}$, and the noise power is unchanged. Covariance matrices $\mathbb{E}[\bar{\mathbf{x}}\bar{\mathbf{x}}^{\mathsf{H}}]$ of signals and of data features are always Hermitian.

Chapter summary

- An $m \times n$ matrix has m rows and n columns; a_{ij} is the entry in row i, column j.
- Matrices of the same size add entry by entry; a scalar multiplies every entry.
- $\mathbf{AB}$ needs the columns of $\mathbf{A}$ to equal the rows of $\mathbf{B}$; $c_{ij} = \sum_k a_{ik} b_{kj}$.

- In general $\mathbf{AB} \neq \mathbf{BA}$.
- $(\mathbf{AB})^\mathsf{T} = \mathbf{B}^\mathsf{T}\mathbf{A}^\mathsf{T}$.
- The trace is the sum of the diagonal entries, and $\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA})$.
- Identity: $\mathbf{A}\mathbf{I}_n = \mathbf{I}_m\mathbf{A} = \mathbf{A}$.
- $\mathbf{A}$ is invertible if $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$; the inverse is unique and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$. The 2×2 inverse uses $1/(ad - bc)$.
- Diagonal and triangular matrices have zeros off, below or above the diagonal.
- Symmetric: $\mathbf{A}^\mathsf{T} = \mathbf{A}$; skew-symmetric: $\mathbf{A}^\mathsf{T} = -\mathbf{A}$.
- Orthogonal: $\mathbf{A}^\mathsf{T}\mathbf{A} = \mathbf{I}$; its columns form an orthonormal set.
- Hermitian: $\mathbf{A} = \mathbf{A}^\mathsf{H}$; unitary: $\mathbf{A}^\mathsf{H}\mathbf{A} = \mathbf{I}$.

Exercises

3.1 Compute $\mathbf{BA}$ for $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}$ and compare it with $\mathbf{AB}$.

3.2 Prove that $\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA})$.

3.3 Find the inverse of $\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$ and verify your answer.

3.4 Show that every square matrix can be written as the sum of a symmetric and a skew-symmetric matrix. *Hint:* consider $\frac{1}{2}(\mathbf{A} + \mathbf{A}^\mathsf{T})$.

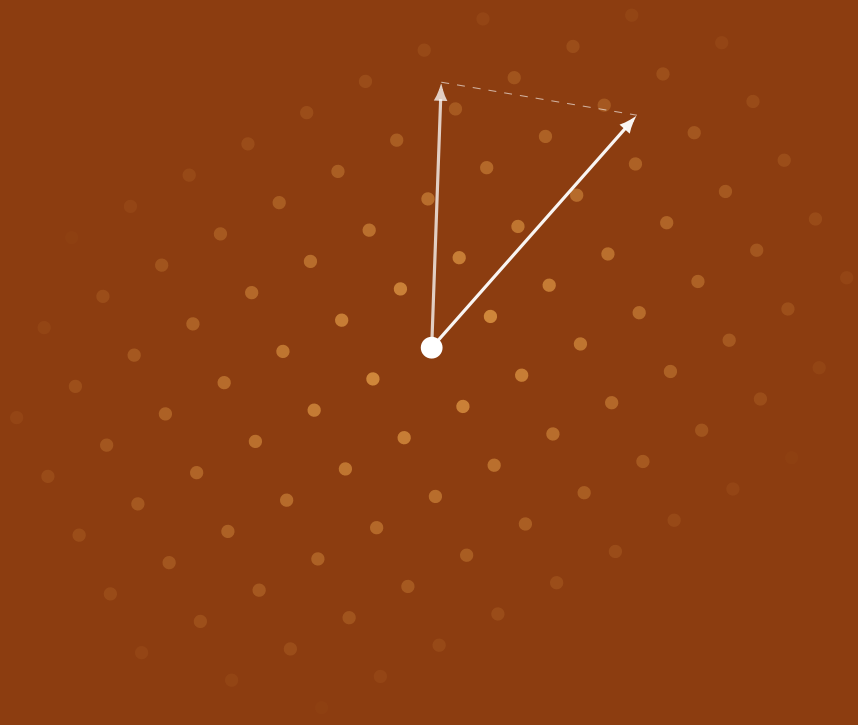
3.5 Verify that $\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ is orthogonal.

3.6 Show that the product of two orthogonal matrices is orthogonal.

3.7 Show that the diagonal entries of a Hermitian matrix are real, and those of a skew-Hermitian matrix are purely imaginary or zero.

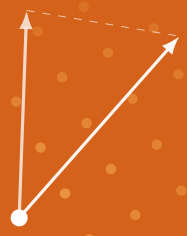
PART III

Equations and Determinants



04

Systems of Linear Equations



Linear systems, Gaussian elimination, echelon and row canonical forms, rank and Gauss–Jordan inversion

In this chapter

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4.1 Linear systems

Definition 4.1 (System of linear equations). A system of m linear equations in n unknowns $x_1, x_2, \ldots, x_n$ is

$$\begin{aligned} L_1 &: \; a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ L_2 &: \; a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ &\vdots \\ L_m &: \; a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m. \end{aligned}$$

It is called an $m \times n$ system, and a **square system** if $m = n$. It is **homogeneous** if all the constants are zero, $b_1 = b_2 = \cdots = b_m = 0$, and **non-homogeneous** otherwise.

Example 4.1 (Checking a solution)

Show that $\begin{bmatrix}2 & -1 & 3 & 1\end{bmatrix}^{\mathsf{T}}$ is a solution of

$$\begin{aligned} L_1 &: \; x_1 + 2x_2 + 3x_3 - x_4 = 8 \\ L_2 &: \; 3x_1 + x_2 - 2x_3 + 4x_4 = 3 \\ L_3 &: \; 2x_1 - x_2 + x_3 + 5x_4 = 13. \end{aligned}$$

Solution. Substituting $x_1 = 2, x_2 = -1, x_3 = 3, x_4 = 1$:

$$\begin{aligned} 2 + 2(-1) + 3(3) - (1) &= 8, \\ 3(2) + (-1) - 2(3) + 4(1) &= 3, \\ 2(2) - (-1) + (3) + 5(1) &= 13. \end{aligned}$$

All three equations hold. ◂

Definition 4.2 (Coefficient and augmented matrices). The $m \times n$ **coefficient matrix** and the $m \times (n+1)$ **augmented matrix** of the system are

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \qquad \mathbf{M} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right].$$

With $\bar{\mathbf{x}} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^{\mathsf{T}}$ and $\bar{\mathbf{b}} = \begin{bmatrix} b_1 & \dots & b_m \end{bmatrix}^{\mathsf{T}}$ the system is

$$\mathbf{A}\bar{\mathbf{x}} = \bar{\mathbf{b}}, \qquad \mathbf{M} = \left[\, \mathbf{A} \mid \bar{\mathbf{b}} \,\right]. \tag{4.1}$$

Example 4.2 (Coefficient and augmented matrices)

For the system of Example 4.1,

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 3 & 1 & -2 & 4 \\ 2 & -1 & 1 & 5 \end{bmatrix}, \qquad \mathbf{M} = \left[\begin{array}{cccc|c} 1 & 2 & 3 & -1 & 8 \\ 3 & 1 & -2 & 4 & 3 \\ 2 & -1 & 1 & 5 & 13 \end{array}\right].$$

$\mathbf{A}$ is 3×4 and $\mathbf{M}$ is 3×5.

4.2 Types of solution

Definition 4.3 (Consistent and inconsistent systems). A system is **consistent** if it has one or more solutions, and **inconsistent** if it has no solution.

Theorem 4.4 (Three possibilities)

A system of linear equations has either a unique solution, no solution, or infinitely many solutions (Figure 4.1).

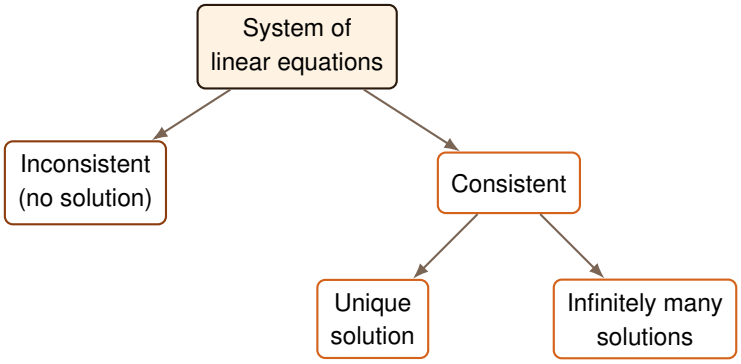


Figure 4.1 A linear system is inconsistent, or consistent with a unique solution or infinitely many solutions.

4.2.1 One equation in one unknown

The simplest system is $ax = b$ ($m = n = 1$).

- If $a \neq 0$, there is the unique solution $x = b/a$.
- If $a = 0$ and $b = 0$, then $ak = 0 = b$ for every scalar k, so there are infinitely many solutions.
- If $a = 0$ but $b \neq 0$, the equation reads $0x = 0 \neq b$, so there is no solution.

Example 4.3 (The three cases for $ax = b$)

Solution.

- $5x - 3 = 2x + 8 \;\Rightarrow\; 3x = 11 \;\Rightarrow\; x = \frac{11}{3}$: a unique solution.
- $7 + 2x - 4 = 3x + 3 - x \;\Rightarrow\; 0x = 0$: every $x = k$ is a solution.
- $3x + 4 - 2x = x - 6 \;\Rightarrow\; 0x = -10$: no solution.

◀

4.2.2 Two equations in two unknowns

A 2×2 system

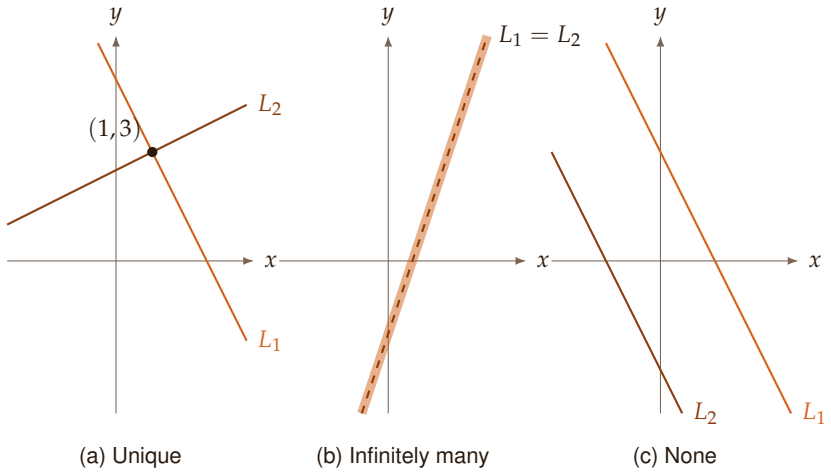
$$a_1x + b_1y = c_1, \qquad a_2x + b_2y = c_2$$

describes two lines in the plane, and its solutions are their common points (Figure 4.2).

- *Unique solution:* the lines meet at one point. This happens when their slopes differ, $-a_1/b_1 \neq -a_2/b_2$, that is, when the coefficients of x and y are not proportional:

$$a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0. \tag{4.2}$$

- *Infinitely many solutions:* the lines coincide (same slope and same intercept), so $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.
- *No solution:* the lines are parallel (same slope, different intercepts), so $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$.



(a) Unique (b) Infinitely many (c) None

Figure 4.2 Two lines: (a) $2x + y = 5$ and $x - 2y = -5$ meet only at $(1, 3)$; (b) $3x - y = 2$ and $6x - 2y = 4$ are the same line; (c) $2x + y = 3$ and $4x + 2y = -6$ are parallel.

Example 4.4 (Classifying 2×2 systems)

Solution.

- $2x + y = 5$, $x - 2y = -5$: $a_1b_2 - a_2b_1 = 2(-2) - 1(1) = -5 \neq 0$, so the solution is unique; it is $(x, y) = (1, 3)$.
- $3x - y = 2$, $6x - 2y = 4$: $\frac{a_1}{a_2} = \frac{3}{6} = \frac{b_1}{b_2} = \frac{-1}{-2} = \frac{c_1}{c_2} = \frac{2}{4} = \frac{1}{2}$, so there are infinitely many solutions.
- $2x + y = 3$, $4x + 2y = -6$: $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{1}{2} \neq \frac{c_1}{c_2} = \frac{3}{-6}$, so there is no solution.

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4.3 Equivalent systems

Definition 4.5 (Linear combination of equations). Multiplying the equations $L_1, \ldots, L_m$ by constants $c_1, \ldots, c_m$ and adding gives the **linear combination**

$$(c_1a_{11} + \cdots + c_ma_{m1})x_1 + \cdots + (c_1a_{1n} + \cdots + c_ma_{mn})x_n = c_1b_1 + \cdots + c_mb_m.$$

Any solution of the original system also satisfies this equation.

Example 4.5 (Linear combination)

For the system of Example 4.1, form $2L_1 - L_2 + 2L_3$.

Solution.

$$\begin{aligned} 2L_1: \;& 2x_1 + 4x_2 + 6x_3 - 2x_4 = 16 \\ -L_2: \;& -3x_1 - x_2 + 2x_3 - 4x_4 = -3 \\ 2L_3: \;& 4x_1 - 2x_2 + 2x_3 + 10x_4 = 26. \end{aligned}$$

Adding gives $3x_1 + x_2 + 10x_3 + 4x_4 = 39$. The solution $\begin{bmatrix} 2 & -1 & 3 & 1 \end{bmatrix}^{\mathsf{T}}$ satisfies it: $6 - 1 + 30 + 4 = 39$. ◂

Definition 4.6 (Equivalent systems). Two systems of linear equations are **equivalent** if they have the same solutions.

Theorem 4.7 (Equivalence by linear combinations)

Two systems have the same solutions if and only if each equation in each system is a linear combination of the equations in the other system.

4.4 Elementary operations and Gaussian elimination

Definition 4.8 (Elementary operations). The **elementary operations** on a system $L_1, \ldots, L_m$ are:

1. *Interchange* two equations: $L_i \leftrightarrow L_j$.
2. *Scale* an equation by a nonzero constant: $kL_i \to L_i$, $k \neq 0$.
3. *Combine*: replace an equation by itself plus a multiple of another, $L_j + kL_i \to L_j$.

Theorem 4.9 (Elementary operations preserve solutions)

Applying an elementary operation to a system gives an equivalent system.

Proof. Each new equation is a linear combination of the old ones. Each operation is also undone by an operation of the same kind ($L_i \leftrightarrow L_j$, $\frac{1}{k}L_i \to L_i$, $L_j - kL_i \to L_j$), so each old equation is a linear combination of the new ones. Theorem 4.7 completes the proof. ■

Gaussian elimination solves a general system in two parts:

- *Forward elimination:* step-by-step reduction to an equivalent, simpler (triangular) system.
- *Backward elimination:* step-by-step back-substitution to solve the simpler system.

The coefficient used to eliminate a variable from the equations below it is called the **pivot**.

Example 4.6 (Gaussian elimination: unique solution)

Solve

$$\begin{aligned} L_1 &: \; x - 2y + 3z = 13 \\ L_2 &: \; 2x - y + 5z = 20 \\ L_3 &: \; -3x + 4y - 7z = -31. \end{aligned}$$

Solution. Use the coefficient 1 of x in L_1 as the pivot to eliminate x:

$$\begin{aligned} -2L_1 + L_2 \to L_2 &: \; 3y - z = -6, \\ 3L_1 + L_3 \to L_3 &: \; -2y + 2z = 8. \end{aligned}$$

Now use the coefficient 3 of y in L_2 as the pivot:

$$\tfrac{2}{3}L_2 + L_3 \to L_3 : \; \tfrac{4}{3}z = 4 \;\Rightarrow\; 4z = 12.$$

The system is now triangular:

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 3 & -1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ -6 \\ 12 \end{bmatrix}.$$

Back-substitution solves for z, y, x in reverse order:

$$z = 3, \qquad 3y - 3 = -6 \Rightarrow y = -1, \qquad x + 2 + 9 = 13 \Rightarrow x = 2.$$

The system has the unique solution $(x, y, z) = (2, -1, 3)$. ◂

Example 4.7 (Gaussian elimination: infinitely many solutions)

Solve

$$\begin{aligned} L_1 &: \; x + 3y - 2z = 2 \\ L_2 &: \; 2x + 7y - 5z = 5 \\ L_3 &: \; 3x + 11y - 8z = 8. \end{aligned}$$

Solution. The operations $-2L_1 + L_2 \to L_2$ and $-3L_1 + L_3 \to L_3$ give

$$L_2 : \; y - z = 1, \qquad L_3 : \; 2y - 2z = 2.$$

L_3 is twice L_2, so it can be dropped, leaving

$$x + 3y - 2z = 2, \qquad y - z = 1.$$

The last equation does not fix y and z. Set the **free variable** $z = a$. Then $y = 1 + a$ and $x = 2 - 3(1 + a) + 2a = -1 - a$. The general solution

$$x = -1 - a, \qquad y = 1 + a, \qquad z = a$$

is valid for every a, so the system has infinitely many solutions. ◂

Example 4.8 (Gaussian elimination: no solution)

Solve

$$\begin{aligned} L_1 &: \; x_1 + x_2 - x_3 + 3x_4 = 3 \\ L_2 &: \; 2x_1 + 5x_2 + x_3 + 4x_4 = 8 \\ L_3 &: \; 3x_1 - 3x_2 - 9x_3 + 13x_4 = 0. \end{aligned}$$

Solution. The operations $-2L_1 + L_2 \to L_2$ and $-3L_1 + L_3 \to L_3$ give

$$L_2 : \; 3x_2 + 3x_3 - 2x_4 = 2, \qquad L_3 : \; -6x_2 - 6x_3 + 4x_4 = -9.$$

Then $2L_2 + L_3 \to L_3$ gives

$$0x_2 + 0x_3 + 0x_4 = -5,$$

which no values satisfy. The system is inconsistent and has no solution. ◂

Remark. Gaussian elimination therefore both reduces a system and tells us which of the three cases of Theorem 4.4 it is in: nonzero pivots on every unknown give a unique solution, a redundant equation leaves a free variable, and an equation $0 = c$ with $c \neq 0$ means no solution.

4.5 Echelon and row canonical forms

The same operations can be applied to the rows of a matrix.

Definition 4.10 (Elementary row operations). For a matrix with rows $R_1, \ldots, R_m$, the **elementary row operations** are

$$R_i \leftrightarrow R_j, \qquad kR_i \to R_i \; (k \neq 0), \qquad kR_i + R_j \to R_j.$$

The last two may be combined as $kR_i + k'R_j \to R_j$ with $k' \neq 0$. Matrices related by row operations are **row equivalent**, written $\mathbf{A} \sim \mathbf{B}$.

Definition 4.11 (Echelon form and pivots). A matrix is in **echelon form** if

1. all zero rows, if any, are at the bottom, and
2. each leading nonzero entry of a row is to the right of the leading nonzero entry of the row above.

The leading nonzero entry of each nonzero row is its **pivot**.

Example 4.9 (An echelon matrix)

The pivots of the echelon matrix below are highlighted:

$$\begin{bmatrix} 0 & 3 & -1 & 2 & 6 & 4 & 0 & 5 \\ 0 & 0 & 0 & 4 & 1 & -2 & 3 & 7 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Definition 4.12 (Row canonical form). A matrix is in **row canonical form** (**reduced row echelon form**) if it is in echelon form and, in addition,

1. each pivot equals 1, and
2. each pivot is the only nonzero entry in its column.

For example, $\begin{bmatrix} 0 & 1 & -2 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{bmatrix}$ is in row canonical form.

Theorem 4.13 (Uniqueness of the row canonical form)

Every matrix $\mathbf{A}$ *is row equivalent to a unique matrix in row canonical form.*

Example 4.10 (Reduction to echelon and row canonical form)

Reduce $\mathbf{A} = \begin{bmatrix} 1 & 3 & -2 & 2 & 1 \\ 2 & 6 & -2 & 10 & 6 \\ 3 & 9 & -3 & 15 & 6 \end{bmatrix}$.

Solution. The operations $-2R_1 + R_2 \to R_2$ and $-3R_1 + R_3 \to R_3$, followed by $-\frac{3}{2}R_2 + R_3 \to R_3$, give the echelon form

$$\mathbf{A} \sim \begin{bmatrix} 1 & 3 & -2 & 2 & 1 \\ 0 & 0 & 2 & 6 & 4 \\ 0 & 0 & 3 & 9 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 & 2 & 1 \\ 0 & 0 & 2 & 6 & 4 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix}.$$

Next, $-\frac{1}{3}R_3 \to R_3$, $-4R_3 + R_2 \to R_2$ and $-R_3 + R_1 \to R_1$ clear the last column; then $\frac{1}{2}R_2 \to R_2$ and $2R_2 + R_1 \to R_1$ clear the third:

$$\mathbf{A} \sim \begin{bmatrix} 1 & 3 & -2 & 2 & 0 \\ 0 & 0 & 2 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 8 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The last matrix is the row canonical form of $\mathbf{A}$. ◂

Definition 4.14 (Rank). The **rank** of a matrix $\mathbf{A}$, written $\operatorname{rank}(\mathbf{A})$, is the number of pivots in an echelon form of $\mathbf{A}$.

For the matrix of Example 4.10, the row canonical form has three pivots, so $\operatorname{rank}(\mathbf{A}) = 3$.

4.6 Solving systems with row operations

An elementary row operation on the augmented matrix $\mathbf{M}$ is the same as the corresponding operation on the system itself. So a system can be solved by row reducing $\mathbf{M} = [\,\mathbf{A} \mid \bar{\mathbf{b}}\,]$.

Example 4.11 (Unique solution)

Solve $x + 2y - z = 6, 2x - y + 3z = -3, 3x + y + z = 4$.

Solution.

$$\mathbf{M} = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 2 & -1 & 3 & -3 \\ 3 & 1 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right].$$

The equivalent system is $\mathbf{I}_3\bar{\mathbf{x}} = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}^{\mathsf{T}}$, so $x = 1, y = 2, z = -1$. Here $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{M}) = n = 3$. ◂

Example 4.12 (Infinitely many solutions)

Solve

$$\begin{aligned} x_1 + 2x_2 - x_3 + 3x_4 &= 4 \\ 2x_1 + 4x_2 - x_3 + 2x_4 &= 5 \\ 3x_1 + 6x_2 - x_3 + x_4 &= 6. \end{aligned}$$

Solution.

$$\mathbf{M} = \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 4 \\ 2 & 4 & -1 & 2 & 5 \\ 3 & 6 & -1 & 1 & 6 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 1 \\ 0 & 0 & 1 & -4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Dropping the zero row gives

$$x_1 + 2x_2 - x_4 = 1, \qquad x_3 - 4x_4 = -3.$$

The pivot variables x_1, x_3 are **basic** (dependent) variables; the others, x_2, x_4, are **free** (independent) variables. The general solution is

$$x_1 = 1 - 2x_2 + x_4, \qquad x_3 = -3 + 4x_4,$$

one solution for each choice of x_2 and x_4. Here $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{M}) = 2 < n = 4$. ◂

Example 4.13 (No solution)

Solve

$$\begin{aligned} x_1 + 2x_2 + x_3 - x_4 &= 3 \\ 2x_1 + 5x_2 + 5x_3 + x_4 &= 2 \\ 3x_1 + 8x_2 + 9x_3 + 3x_4 &= -5. \end{aligned}$$

Solution.

$$\mathbf{M} = \left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 3 \\ 2 & 5 & 5 & 1 & 2 \\ 3 & 8 & 9 & 3 & -5 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 3 \\ 0 & 1 & 3 & 3 & -4 \\ 0 & 0 & 0 & 0 & -6 \end{array}\right].$$

The last row is the inconsistent equation $0x_1 + 0x_2 + 0x_3 + 0x_4 = -6$, so there is no solution. Here $\operatorname{rank}(\mathbf{A}) = 2 < \operatorname{rank}(\mathbf{M}) = 3$. ◂

The three examples illustrate the following theorem.

Theorem 4.15 (Existence and uniqueness)

Consider a system $\mathbf{A}\bar{\mathbf{x}} = \bar{\mathbf{b}}$ in n unknowns with augmented matrix $\mathbf{M} = [\,\mathbf{A} \mid \bar{\mathbf{b}}\,]$.

1. *The system has no solution if and only if $\operatorname{rank}(\mathbf{A}) < \operatorname{rank}(\mathbf{M})$.*
2. *It has a solution if and only if $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{M})$.*
3. *The solution is unique if $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{M}) = n$.*
4. *There are infinitely many solutions if and only if $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{M}) < n$.*

Case	Rank condition	Sign in echelon form
No solution	$\operatorname{rank}(\mathbf{A}) < \operatorname{rank}(\mathbf{M})$	a row $[\,0 \cdots 0 \mid c\,]$, $c \neq 0$
Unique solution	$\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{M}) = n$	a pivot in every column of $\mathbf{A}$
Infinitely many	$\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{M}) < n$	at least one free variable

Table 4.1 The rank test for $\mathbf{A}\bar{\mathbf{x}} = \bar{\mathbf{b}}$ with n unknowns.

4.7 Square systems and the inverse

In a square system the number of equations equals the number of unknowns:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \qquad \text{that is,} \qquad \mathbf{A}\bar{\mathbf{x}} = \bar{\mathbf{b}}.$$

Such a system has a unique solution only if $\mathbf{A}$ is invertible. Multiplying both sides by $\mathbf{A}^{-1}$ gives $\mathbf{A}^{-1}\mathbf{A}\bar{\mathbf{x}} = \mathbf{A}^{-1}\bar{\mathbf{b}}$, so

$$\bar{\mathbf{x}} = \mathbf{A}^{-1}\bar{\mathbf{b}}. \tag{4.3}$$

Example 4.14 (Solving with the inverse)

Solve $x + y + 2z = 3$, $x + 2y + 4z = 4$, $2x + 3y + 7z = 8$, given that

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 4 \\ 2 & 3 & 7 \end{bmatrix}, \qquad \mathbf{A}^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & -2 \\ -1 & -1 & 1 \end{bmatrix}.$$

Solution. By (4.3),

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & -2 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

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Theorem 4.16 (Gauss–Jordan inversion)

To invert an $n \times n$ matrix $\mathbf{A}$, form the $n \times 2n$ block matrix $\mathbf{M} = [\,\mathbf{A} \mid \mathbf{I}\,]$ and row reduce it to row canonical form. If the result is $[\,\mathbf{I} \mid \mathbf{B}\,]$, then $\mathbf{A}^{-1} = \mathbf{B}$. If the left block cannot be reduced to $\mathbf{I}$, then $\mathbf{A}$ is not invertible.

Example 4.15 (Gauss–Jordan inversion)

Find the inverse of $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$.

Solution.

$$\mathbf{M} = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 2 & -1 \\ 0 & 1 & 0 & -5 & 3 & -1 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right].$$

Therefore

$$\mathbf{A}^{-1} = \begin{bmatrix} -2 & 2 & -1 \\ -5 & 3 & -1 \\ 3 & -2 & 1 \end{bmatrix}.$$

◀

AIML & 6G Connection

A MIMO receiver observing $\bar{\mathbf{y}} = \mathbf{H}\bar{\mathbf{x}} + \bar{\mathbf{n}}$ with a square, invertible channel can recover the symbols by *zero-forcing*, $\hat{\bar{\mathbf{x}}} = \mathbf{H}^{-1}\bar{\mathbf{y}}$: a linear system solved at every symbol time. In machine learning, the normal equations $(\mathbf{X}^{\mathsf{T}}\mathbf{X})\bar{\mathbf{w}} = \mathbf{X}^{\mathsf{T}}\bar{\mathbf{y}}$ of linear regression are a square linear system. In practice both are solved by Gaussian elimination (or its factorised form, LU decomposition) rather than by forming an explicit inverse, at a cost of about $\frac{2}{3}n^3$ operations.

Chapter summary

- An $m \times n$ system has m equations in n unknowns; it is square if $m = n$.
- The coefficient matrix $\mathbf{A}$ holds the coefficients; the augmented matrix $\mathbf{M} = [\,\mathbf{A} \mid \bar{\mathbf{b}}\,]$ adds the constants.
- A system is consistent (one or more solutions) or inconsistent (no solution); it has a unique solution, no solution or infinitely many.

- Two lines have a unique intersection if $a_1b_2 - a_2b_1 \neq 0$.
- Equivalent systems have the same solutions; elementary operations $L_i \leftrightarrow L_j$, $kL_i \to L_i$, $L_j + kL_i \to L_j$ preserve them.
- Gaussian elimination: forward elimination, then back-substitution.
- Echelon form: zero rows at the bottom, pivots move right. Row canonical form: pivots equal 1 and are alone in their columns.
- $\text{rank}(\mathbf{A})$ is the number of pivots in an echelon form.
- A solution exists iff $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{M})$; it is unique iff this also equals n.
- Row reducing $[\,\mathbf{A} \mid \mathbf{I}\,]$ to $[\,\mathbf{I} \mid \mathbf{B}\,]$ gives $\mathbf{A}^{-1} = \mathbf{B}$.

Exercises

4.1 Solve by Gaussian elimination: $x + y + z = 6$, $2x - y + z = 3$, $x + 2y - z = 2$.

4.2 Find the general solution of $x + 2y - 3z = 1$, $2x + 5y - 8z = 4$, $3x + 8y - 13z = 7$, and identify the free variable.

4.3 Show that $x + y - z = 1$, $x + 2y - 3z = 2$, $2x + 3y - 4z = 5$ has no solution by comparing $\text{rank}(\mathbf{A})$ and $\text{rank}(\mathbf{M})$.

4.4 For which values of k does the system $x + ky = 1$, $kx + y = 1$ have (a) a unique solution, (b) no solution, (c) infinitely many solutions?

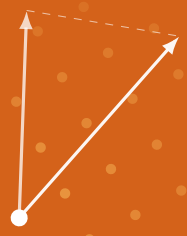
4.5 Find the rank of the 5×8 echelon matrix in this chapter, and the rank of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 0 & 1 \end{bmatrix}$.

4.6 Use Gauss–Jordan inversion to find the inverse of $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$.

4.7 Show that a homogeneous system $\mathbf{A}\bar{\mathbf{x}} = \bar{\mathbf{0}}$ is always consistent, and that it has infinitely many solutions when it has more unknowns than equations.

05

Determinants



Determinants: definition, cofactors, properties, invertibility, adjoint, Cramer's rule, block matrices and volume

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5.1 The determinant

Definition 5.1 (Determinant). Each $n \times n$ square matrix $\mathbf{A} = [a_{ij}]$ is assigned a special scalar called the **determinant** of $\mathbf{A}$, denoted by $\det(\mathbf{A})$ or $|\mathbf{A}|$:

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

Definition 5.2 (Determinants of order 1 and 2). The determinant of order 1 is the scalar itself, $\det(a_{11}) = a_{11}$. The determinant of order 2 is

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}. \tag{5.1}$$

The determinant of order 2 is easily remembered with the diagram in Figure 5.1: it equals the product of the entries along the plus-labelled arrow minus the product of the entries along the minus-labelled arrow.

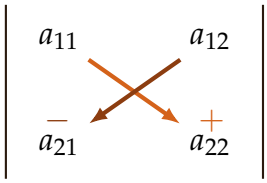


Figure 5.1 Order-2 determinant: the product along the + diagonal minus the product along the − diagonal.

Example 5.1 (Determinants of order 1 and 2)

Solution. Order 1:

$$\det(15) = 15, \qquad \det(-9) = -9, \qquad \det(t+5) = t+5.$$

Order 2:

$$\begin{vmatrix} 7 & 2 \\ 3 & 4 \end{vmatrix} = 7 \times 4 - 2 \times 3 = 28 - 6 = 22, \qquad \begin{vmatrix} 6 & -1 \\ 3 & 2 \end{vmatrix} = 12 + 3 = 15.$$

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Remark. For a 1×1 matrix, $\det(-9) = -9$: the bars in $|\mathbf{A}|$ denote the determinant, not an absolute value.

5.1.1 Application to two linear equations

Consider two linear equations in two unknowns,

$$a_1x + b_1y = c_1, \qquad a_2x + b_2y = c_2,$$

and let

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

The system has a unique solution if and only if $D \neq 0$. The solution can then be expressed in terms of determinants as

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}. \tag{5.2}$$

Example 5.2 (Two equations by determinants)

Solve $3x + 2y = 8, 5x - 4y = 6$.

Solution. By (5.2),

$$x = \frac{\begin{vmatrix} 8 & 2 \\ 6 & -4 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 5 & -4 \end{vmatrix}} = \frac{-44}{-22} = 2, \qquad y = \frac{\begin{vmatrix} 3 & 8 \\ 5 & 6 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 5 & -4 \end{vmatrix}} = \frac{-22}{-22} = 1.$$

◀

5.2 Minors and cofactors

Definition 5.3 (Minor and cofactor). Let $\mathbf{A}$ be an $n \times n$ matrix, and let $\mathbf{M}_{ij}$ denote the $(n-1) \times (n-1)$ submatrix of $\mathbf{A}$ obtained by deleting its ith row and jth column. The determinant $|\mathbf{M}_{ij}|$ is called the **minor** of the element a_{ij}. The **cofactor** of a_{ij}, denoted C_{ij}, is the signed minor

$$C_{ij} = (-1)^{i+j} |\mathbf{M}_{ij}|. \tag{5.3}$$

Example 5.3 (Minor and cofactor)

For $\mathbf{A} = \begin{bmatrix} 2 & -1 & 4 \\ 3 & 5 & 1 \\ -2 & 6 & 7 \end{bmatrix}$, find $|\mathbf{M}_{23}|$ and C_{23}.

Solution. Deleting row 2 and column 3,

$$|\mathbf{M}_{23}| = \begin{vmatrix} 2 & -1 \\ -2 & 6 \end{vmatrix} = 12 - 2 = 10, \qquad C_{23} = (-1)^{2+3} \times 10 = -10.$$

◀

Theorem 5.4 (Laplace expansion)

The determinant of a square matrix $\mathbf{A} = [a_{ij}]$ equals the sum of the products obtained by multiplying the elements of any row (column) by their respective cofactors. Along row i and along column j,

$$|\mathbf{A}| = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij},$$

$$|\mathbf{A}| = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{i=1}^{n} a_{ij}C_{ij}.$$

Example 5.4 (Laplace expansion)

Evaluate $\begin{vmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \\ 1 & 3 & 2 \end{vmatrix}$.

Solution. Expanding along the first row,

$$2\begin{vmatrix} -2 & 4 \\ 3 & 2 \end{vmatrix} - 1\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix} = -32 - 2 - 11 = -45.$$

◀

5.3 Properties of the determinant

Throughout, $\mathbf{A}$ is a square matrix. Each property stated for rows holds equally for columns.

Theorem 5.5 (Basic properties)

1. Transpose: $|\mathbf{A}| = |\mathbf{A}^{\mathsf{T}}|$.
2. Zero row: *if* $\mathbf{A}$ *has a row (column) of zeros, then* $|\mathbf{A}| = 0$.
3. Identical rows: *if* $\mathbf{A}$ *has two identical rows (columns), then* $|\mathbf{A}| = 0$.
4. Triangular or diagonal: *if* $\mathbf{A}$ *is lower triangular, upper triangular or diagonal, then* $|\mathbf{A}|$ *is the product of the diagonal elements,*

$$|\mathbf{A}| = a_{11} \times a_{22} \times \cdots \times a_{nn}.$$

In particular, $|\mathbf{I}| = 1 \times 1 \times \cdots \times 1 = 1$.

5.3.1 Effect of row operations

Theorem 5.6 (Row operations and the determinant)

Let $\mathbf{B}$ *be obtained from* $\mathbf{A}$ *by a row (column) operation.*

1. Interchange: *if two rows (columns) are interchanged,* $|\mathbf{B}| = -|\mathbf{A}|$.
2. Scalar multiple: *if a row (column) is multiplied by a scalar* k, $|\mathbf{B}| = k|\mathbf{A}|$.
3. Row operation: *if a multiple of one row (column) is added to another,* $R_i + kR_j \to R_i$, *then* $|\mathbf{B}| = |\mathbf{A}|$.

Proof of (2). Scale the jth column by k and expand along that column. The cofactors C_{ij} do not involve column j, so the determinant becomes

$$ka_{1j}C_{1j} + ka_{2j}C_{2j} + \cdots + ka_{nj}C_{nj} = \sum_{i=1}^{n} ka_{ij}C_{ij} = k|\mathbf{A}|.$$

■

The next three examples start from Example 5.4, where

$$\begin{vmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \\ 1 & 3 & 2 \end{vmatrix} = -45.$$

Example 5.5 (Interchange)

Interchange the first two columns and recompute the determinant.

Solution.

$$\begin{vmatrix} 1 & 2 & -1 \\ -2 & 3 & 4 \\ 3 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} -2 & 4 \\ 3 & 2 \end{vmatrix} + (-1)\begin{vmatrix} -2 & 3 \\ 3 & 1 \end{vmatrix} = 2 + 32 + 11 = 45.$$

The sign has changed, as Theorem 5.6(1) predicts. ◂

Example 5.6 (Scalar multiple)

Multiply the first row by 2 and recompute the determinant.

Solution.

$$\begin{vmatrix} 4 & 2 & -2 \\ 3 & -2 & 4 \\ 1 & 3 & 2 \end{vmatrix} = 4\begin{vmatrix} -2 & 4 \\ 3 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} + (-2)\begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix}$$

$$= -64 - 4 - 22 = -90 = -45 \times 2.$$

◂

Example 5.7 (Adding a multiple of a row)

Perform $R_1 + 2R_2 \to R_1$ and recompute the determinant.

Solution.

$$\begin{vmatrix} 8 & -3 & 7 \\ 3 & -2 & 4 \\ 1 & 3 & 2 \end{vmatrix} = 8\begin{vmatrix} -2 & 4 \\ 3 & 2 \end{vmatrix} + 3\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} + 7\begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix} = -128 + 6 + 77 = -45.$$

The determinant is unchanged. ◂

5.3.2 Sum and product properties

Theorem 5.7 (Sum property)

The determinant behaves like a linear function of each row (column) of the matrix. For example,

$$\begin{vmatrix} a_{11} + a'_{11} & a_{12} & a_{13} \\ a_{21} + a'_{21} & a_{22} & a_{23} \\ a_{31} + a'_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a'_{11} & a_{12} & a_{13} \\ a'_{21} & a_{22} & a_{23} \\ a'_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Proof. Expand along the row (column) that is split. Since the cofactors C_{ij} do not depend on that row,

$$|\mathbf{A}| = \sum_{j=1}^{n} \left(a_{ij} + a'_{ij}\right) C_{ij} = \sum_{j=1}^{n} a_{ij}C_{ij} + \sum_{j=1}^{n} a'_{ij}C_{ij}.$$

■

Theorem 5.8 (Product property)

For square matrices $\mathbf{A}$ *and* $\mathbf{B}$ *of the same size,*

$$|\mathbf{AB}| = |\mathbf{A}|\,|\mathbf{B}|. \tag{5.4}$$

Example 5.8 (Determinant of an orthogonal matrix)

Let $\mathbf{U}$ be an orthogonal matrix. Show that $|\mathbf{U}| = \pm 1$.

Solution. Since $\mathbf{U}\mathbf{U}^{\mathsf{T}} = \mathbf{I}$, Theorem 5.8 and $|\mathbf{U}^{\mathsf{T}}| = |\mathbf{U}|$ give

$$|\mathbf{U}\mathbf{U}^{\mathsf{T}}| = |\mathbf{I}| = 1 \;\Rightarrow\; |\mathbf{U}|\,|\mathbf{U}^{\mathsf{T}}| = 1 \;\Rightarrow\; |\mathbf{U}|^2 = 1 \;\Rightarrow\; |\mathbf{U}| = \pm 1.$$

◂

5.3.3 Dependent rows

Theorem 5.9 (Dependent rows give a zero determinant)

1. *If one row is a scalar multiple of another,* $R_i = \alpha R_j$*, then* $|\mathbf{A}| = 0$*. The same holds for columns.*
2. *More generally, if a row (column) is a linear combination of other rows (columns),*

$$R_i = \alpha_1 R_{j_1} + \alpha_2 R_{j_2} + \cdots + \alpha_p R_{j_p},$$

then $|\mathbf{A}| = 0$.

Proof. (1) The row operation $R_i - \alpha R_j \to R_i$ makes row i zero and does not change the determinant; a matrix with a zero row has determinant zero. (2) Similarly, the row (column) operation

$$R_i - \left(\alpha_1 R_{j_1} + \alpha_2 R_{j_2} + \cdots + \alpha_p R_{j_p}\right) \to R_i$$

makes row i zero without changing the determinant, so $|\mathbf{A}| = 0$. ■

Operation on $\mathbf{A}$	Effect on $\lvert\mathbf{A}\rvert$
Interchange two rows, $R_i \leftrightarrow R_j$	multiplied by -1
Scale a row, $kR_i \to R_i$	multiplied by k
Add a multiple of a row, $R_i + kR_j \to R_i$	unchanged
Transpose, $\mathbf{A}^\mathsf{T}$	unchanged
Product with $\mathbf{B}$, $\mathbf{AB}$	multiplied by $\lvert\mathbf{B}\rvert$

Table 5.1 How operations affect the determinant.

5.4 Invertibility and the determinant

Theorem 5.10 (Invertibility test)

A square matrix $\mathbf{A}$ *is invertible if and only if* $|\mathbf{A}| \neq 0$.

Proof. By Theorem 5.6, each elementary row operation only multiplies the determinant by a nonzero constant:

- $\alpha R_i \to R_i$ multiplies it by $\alpha \neq 0$;
- $R_i + \alpha R_j \to R_i$ leaves it unchanged (multiplies it by 1);
- $R_i \leftrightarrow R_j$ multiplies it by -1.

Row reduce $\mathbf{A}$ to its row canonical form, and let $\beta \neq 0$ be the product of all the constants that multiply $|\mathbf{A}|$ along the way.

Invertible. An invertible $\mathbf{A}$ can be reduced to $\mathbf{I}$ by row operations, so

$$\beta|\mathbf{A}| = |\mathbf{I}| = 1 \;\Rightarrow\; |\mathbf{A}| = \frac{1}{\beta} \neq 0.$$

Non-invertible. If $\mathbf{A}$ is not invertible, its row canonical form $\mathbf{A}_{\text{eq}}$ contains at least one row of zeros. Therefore

$$|\mathbf{A}_{\text{eq}}| = 0 \;\Rightarrow\; \beta|\mathbf{A}| = 0 \;\Rightarrow\; |\mathbf{A}| = 0,$$

since $\beta \neq 0$. ∎

5.5 The classical adjoint and the inverse

Definition 5.11 (Classical adjoint). Let C_{ij} denote the cofactor of a_{ij}. The **classical adjoint** (or **adjugate**) of $\mathbf{A}$ is the transpose of the matrix of cofactors:

$$\operatorname{adj}(\mathbf{A}) = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}.$$

Theorem 5.12 (Adjoint property)

For any square matrix $\mathbf{A}$*,*

$$\mathbf{A}\,\mathrm{adj}(\mathbf{A}) = \mathrm{adj}(\mathbf{A})\mathbf{A} = |\mathbf{A}|\mathbf{I}. \tag{5.5}$$

Proof. Let $\mathbf{B} = \mathbf{A}\,\mathrm{adj}(\mathbf{A})$:

$$\mathbf{B} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}.$$

A diagonal entry is the Laplace expansion of $|\mathbf{A}|$ along row i:

$$b_{ii} = \sum_{k=1}^{n} a_{ik}C_{ik} = |\mathbf{A}|.$$

For $i \neq j$,

$$b_{ij} = \sum_{k=1}^{n} a_{ik}C_{jk} = 0.$$

This sum is the determinant of the matrix obtained from $\mathbf{A}$ by replacing its jth row by its ith row. Since two rows become equal, that determinant is zero. Therefore

$$\mathbf{B} = \begin{bmatrix} |\mathbf{A}| & 0 & \dots & 0 \\ 0 & |\mathbf{A}| & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & |\mathbf{A}| \end{bmatrix} = |\mathbf{A}|\mathbf{I}.$$

The same argument with columns gives $\mathrm{adj}(\mathbf{A})\mathbf{A} = |\mathbf{A}|\mathbf{I}$. ∎

When $|\mathbf{A}| \neq 0$, dividing (5.5) by $|\mathbf{A}|$ gives $\mathbf{A}\,\dfrac{\mathrm{adj}(\mathbf{A})}{|\mathbf{A}|} = \mathbf{I}$, so

$$\mathbf{A}^{-1} = \frac{\mathrm{adj}(\mathbf{A})}{|\mathbf{A}|}. \tag{5.6}$$

Example 5.9 (Inverse via the adjoint)

Find the inverse of $\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ -2 & 1 & 4 \end{bmatrix}$.

Solution. The cofactors include, for example,

$$C_{11} = \begin{vmatrix} 0 & 2 \\ 1 & 4 \end{vmatrix} = 0 - 2 = -2, \qquad C_{21} = -\begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} = (-1)(8+1) = -9.$$

Computing all nine and transposing gives

$$\operatorname{adj}(\mathbf{A}) = \begin{bmatrix} -2 & -9 & 4 \\ -16 & 2 & -5 \\ 3 & -5 & -6 \end{bmatrix}.$$

Expanding along row 2,

$$|\mathbf{A}| = -3(8+1) - 2(1+4) = -27 - 10 = -37.$$

Hence

$$\mathbf{A}^{-1} = \frac{\operatorname{adj}(\mathbf{A})}{|\mathbf{A}|} = -\frac{1}{37}\begin{bmatrix} -2 & -9 & 4 \\ -16 & 2 & -5 \\ 3 & -5 & -6 \end{bmatrix}.$$

◀

5.6 Cramer's rule

Consider the square system $\mathbf{A}\bar{\mathbf{x}} = \bar{\mathbf{b}}$,

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Let $\mathbf{A}_i$ be the matrix obtained from $\mathbf{A}$ by replacing its ith column by the column vector $\bar{\mathbf{b}}$. For example,

$$\mathbf{A}_2 = \begin{bmatrix} a_{11} & b_1 & \dots & a_{1n} \\ a_{21} & b_2 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & b_n & \dots & a_{nn} \end{bmatrix}.$$

Theorem 5.13 (Cramer's rule)

The system $\mathbf{A}\bar{\mathbf{x}} = \bar{\mathbf{b}}$ has a unique solution if and only if $|\mathbf{A}| \neq 0$. The unique solution is

$$x_1 = \frac{|\mathbf{A}_1|}{|\mathbf{A}|}, \quad x_2 = \frac{|\mathbf{A}_2|}{|\mathbf{A}|}, \quad \dots, \quad x_n = \frac{|\mathbf{A}_n|}{|\mathbf{A}|}. \tag{5.7}$$

Example 5.10 (Cramer's rule)

Solve

$$\begin{aligned} x + 2y + z &= 3 \\ 2x - y + 3z &= 14 \\ 3x + y - 2z &= -1. \end{aligned}$$

Solution. The coefficient matrix and its determinant are

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 3 \\ 3 & 1 & -2 \end{bmatrix}, \qquad |\mathbf{A}| = 1(2-3) - 2(-4-9) + 1(2+3) = -1 + 26 + 5 = 30.$$

Replacing the columns of $\mathbf{A}$ in turn by $\bar{\mathbf{b}} = \begin{bmatrix} 3 & 14 & -1 \end{bmatrix}^{\mathsf{T}}$:

$$\mathbf{A}_1 = \begin{bmatrix} 3 & 2 & 1 \\ 14 & -1 & 3 \\ -1 & 1 & -2 \end{bmatrix}, \qquad \mathbf{A}_2 = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 14 & 3 \\ 3 & -1 & -2 \end{bmatrix}, \qquad \mathbf{A}_3 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 14 \\ 3 & 1 & -1 \end{bmatrix},$$

with determinants

$$\begin{aligned} |\mathbf{A}_1| &= 3(2-3) - 2(-28+3) + 1(14-1) = -3 + 50 + 13 = 60, \\ |\mathbf{A}_2| &= 1(-28+3) - 3(-4-9) + 1(-2-42) = -25 + 39 - 44 = -30, \\ |\mathbf{A}_3| &= 1(1-14) - 2(-2-42) + 3(2+3) = -13 + 88 + 15 = 90. \end{aligned}$$

Therefore the unique solution is

$$x = \frac{|\mathbf{A}_1|}{|\mathbf{A}|} = \frac{60}{30} = 2, \qquad y = \frac{|\mathbf{A}_2|}{|\mathbf{A}|} = \frac{-30}{30} = -1, \qquad z = \frac{|\mathbf{A}_3|}{|\mathbf{A}|} = \frac{90}{30} = 3.$$

◀

5.7 Principal minors

Definition 5.14 (Principal minor). A minor is **principal** if its row and column indices are the same, or equivalently, if the diagonal elements of the minor come from the diagonal of the matrix.

The sign $(-1)^{i+j}$ attached to a principal minor is always $+1$, because the sum of a row subscript and the identical column subscript is always even.

Example 5.11 (Principal minors)

Find the principal minors of orders 1, 2 and 3 of $\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \\ 3 & 0 & 1 \end{bmatrix}$.

Solution. The principal minors of order 1 are the diagonal entries 2, 4 and 1. There are three principal minors of order 2:

$$\begin{vmatrix} 4 & -2 \\ 0 & 1 \end{vmatrix} = 4, \qquad \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = -7, \qquad \begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} = 9.$$

There is only one principal minor of order 3, the determinant of $\mathbf{A}$ itself:

$$\begin{vmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \\ 3 & 0 & 1 \end{vmatrix} = 2(4-0)+1(1+6)+3(0-12) = 8+7-36 = -21.$$

◀

5.8 Block matrices

Theorem 5.15 (Triangular block matrix)

Suppose $\mathbf{M}$ *is an upper (lower) triangular block matrix with diagonal blocks* $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$*. Then*

$$|\mathbf{M}| = |\mathbf{A}_1|\,|\mathbf{A}_2| \cdots |\mathbf{A}_n|. \tag{5.8}$$

Example 5.12 (Block determinant)

Find $|\mathbf{M}|$ for

$$\mathbf{M} = \left[\begin{array}{cc|ccc} 3 & 1 & 2 & -1 & 4 \\ 2 & 4 & 0 & 5 & 1 \\ \hline 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & -2 & 1 & 3 \\ 0 & 0 & 4 & 0 & 1 \end{array}\right] = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_3 \\ \mathbf{0} & \mathbf{A}_2 \end{bmatrix}.$$

Solution. $\mathbf{M}$ is an upper triangular block matrix. Its diagonal blocks have

$$|\mathbf{A}_1| = \begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} = 12 - 2 = 10,$$

$$|\mathbf{A}_2| = \begin{vmatrix} 1 & 3 & 2 \\ -2 & 1 & 3 \\ 4 & 0 & 1 \end{vmatrix} = 1(1-0) - 3(-2-12) + 2(0-4) = 1 + 42 - 8 = 35.$$

By Theorem 5.15,

$$|\mathbf{M}| = \begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} \begin{vmatrix} 1 & 3 & 2 \\ -2 & 1 & 3 \\ 4 & 0 & 1 \end{vmatrix} = 10 \times 35 = 350.$$

◀

Remark. If $\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$, where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are square matrices, it is *not* generally true that $|\mathbf{M}| = |\mathbf{A}||\mathbf{D}| - |\mathbf{B}||\mathbf{C}|$.

5.9 Determinant and volume

Determinants are related to area and volume as follows. Let $\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \ldots, \bar{\mathbf{u}}_n$ in $\mathbb{R}^n$ determine a region S. When $n = 2$, S is a parallelogram; when $n = 3$, S is a parallelepiped.

Theorem 5.16 (Volume as a determinant)

Let $V(S)$ denote the volume of S, and let $\mathbf{A}$ be the matrix with columns $\bar{\mathbf{u}}_1, \ldots, \bar{\mathbf{u}}_n$:

$$\mathbf{A} = \begin{bmatrix} | & | & & | \\ \bar{\mathbf{u}}_1 & \bar{\mathbf{u}}_2 & \cdots & \bar{\mathbf{u}}_n \\ | & | & & | \end{bmatrix}.$$

Then

$$V(S) = |\det(\mathbf{A})|. \tag{5.9}$$

Example 5.13 (Volume of a parallelepiped)

Find the volume $V(S)$ of the parallelepiped S in $\mathbb{R}^3$ determined by

$$\bar{\mathbf{u}}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \qquad \bar{\mathbf{u}}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \qquad \bar{\mathbf{u}}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Solution. The parallelepiped is shown in Figure 5.2. Its volume is the absolute value of the determinant of the matrix with columns $\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \bar{\mathbf{u}}_3$:

$$V(S) = \left|\begin{vmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 1 & 0 & 2 \end{vmatrix}\right| = |2(6-0) - 1(0-1) + 0(0-3)| = |12+1| = 13.$$

◀

AIML & 6G Connection

Because $|\det(\mathbf{A})|$ is the factor by which $\mathbf{A}$ scales volume, determinants appear wherever probability densities are transformed. The Gaussian density uses $\det(\blacksquare)$, and normalizing-flow generative models track $\log|\det \mathbf{J}|$ of each layer's Jacobian. In wireless, the capacity of a MIMO channel is $\log_2 \det\left(\mathbf{I} + \frac{\rho}{N_t}\mathbf{H}\mathbf{H}^{\mathsf{H}}\right)$ bits/s/Hz.

5.10 Complex matrices

The determinant of a complex matrix is computed in the same way as that of a real matrix. It satisfies

$$|\mathbf{A}^{\mathsf{H}}| = |(\mathbf{A}^*)^{\mathsf{T}}| = |\mathbf{A}^*| = |\mathbf{A}|^*. \tag{5.10}$$

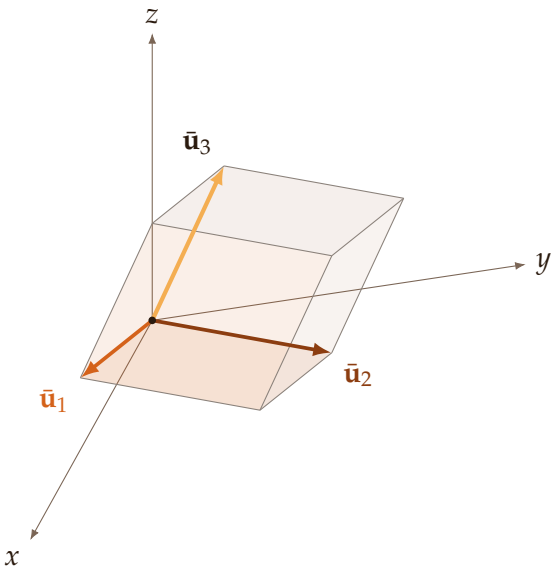


Figure 5.2 The parallelepiped in $\mathbb{R}^3$ spanned by $\bar{\mathbf{u}}_1 = (2,0,1)$, $\bar{\mathbf{u}}_2 = (1,3,0)$ and $\bar{\mathbf{u}}_3 = (0,1,2)$. Its volume is 13.

Example 5.14 (Determinant of a unitary matrix)

Let $\mathbf{U}$ be unitary. Show that $|\det \mathbf{U}| = 1$.

Solution. A unitary matrix satisfies $\mathbf{U}\mathbf{U}^{\mathsf{H}} = \mathbf{I}$. Using (5.4) and (5.10),

$$|\mathbf{U}|\,|\mathbf{U}^{\mathsf{H}}| = |\mathbf{I}| = 1 \;\Rightarrow\; |\mathbf{U}|\,|\mathbf{U}|^{*} = 1 \;\Rightarrow\; |\det \mathbf{U}|^2 = 1 \;\Rightarrow\; |\det \mathbf{U}| = 1.$$

Hence $\det \mathbf{U} = e^{j\theta}$ for some real θ: it lies on the unit circle. ◂

5.11 Some determinant problems

We now work three problems that use the properties of this chapter.

Example 5.15 (Problem 1)

Simplify $\begin{vmatrix} a & a^2 & b+c \\ b & b^2 & c+a \\ c & c^2 & a+b \end{vmatrix}$.

Solution. Perform $C_1 + C_3 \to C_3$ and take out the common factor of the third column:

$$\begin{vmatrix} a & a^2 & b+c \\ b & b^2 & c+a \\ c & c^2 & a+b \end{vmatrix} = \begin{vmatrix} a & a^2 & a+b+c \\ b & b^2 & b+c+a \\ c & c^2 & c+a+b \end{vmatrix} = (a+b+c)\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}.$$

Now perform $R_1 - R_2 \to R_1$ and $R_2 - R_3 \to R_2$, and expand along the third column:

$$\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = \begin{vmatrix} a-b & a^2-b^2 & 0 \\ b-c & b^2-c^2 & 0 \\ c & c^2 & 1 \end{vmatrix}$$
$$= (a-b)(b^2-c^2) - (b-c)(a^2-b^2)$$
$$= (a-b)(b-c)(b+c-a-b)$$
$$= (a-b)(b-c)(c-a).$$

Therefore

$$\begin{vmatrix} a & a^2 & b+c \\ b & b^2 & c+a \\ c & c^2 & a+b \end{vmatrix} = (a+b+c)\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = (a+b+c)(a-b)(b-c)(c-a).$$

◀

Example 5.16 (Problem 2)

Simplify $\begin{vmatrix} 1+a & 1+ax & 1+ax^2 \\ 1+b & 1+bx & 1+bx^2 \\ 1+c & 1+cx & 1+cx^2 \end{vmatrix}$.

Solution. Using the sum property (Theorem 5.7) on the first column,

$$\begin{vmatrix} 1+a & 1+ax & 1+ax^2 \\ 1+b & 1+bx & 1+bx^2 \\ 1+c & 1+cx & 1+cx^2 \end{vmatrix} = \underbrace{\begin{vmatrix} 1 & 1+ax & 1+ax^2 \\ 1 & 1+bx & 1+bx^2 \\ 1 & 1+cx & 1+cx^2 \end{vmatrix}}_{D_1} + \underbrace{\begin{vmatrix} a & 1+ax & 1+ax^2 \\ b & 1+bx & 1+bx^2 \\ c & 1+cx & 1+cx^2 \end{vmatrix}}_{D_2}.$$

Simplifying D_1. Split the second column, then the third. Each determinant with two equal columns is zero:

$$D_1 = \underbrace{\begin{vmatrix} 1 & 1 & 1+ax^2 \\ 1 & 1 & 1+bx^2 \\ 1 & 1 & 1+cx^2 \end{vmatrix}}_{0} + \begin{vmatrix} 1 & ax & 1+ax^2 \\ 1 & bx & 1+bx^2 \\ 1 & cx & 1+cx^2 \end{vmatrix}$$
$$= \underbrace{\begin{vmatrix} 1 & ax & 1 \\ 1 & bx & 1 \\ 1 & cx & 1 \end{vmatrix}}_{0} + \begin{vmatrix} 1 & ax & ax^2 \\ 1 & bx & bx^2 \\ 1 & cx & cx^2 \end{vmatrix}.$$

Performing $C_3 - xC_2 \to C_3$, the last determinant reduces to

$$\begin{vmatrix} 1 & ax & ax^2 \\ 1 & bx & bx^2 \\ 1 & cx & cx^2 \end{vmatrix} = \begin{vmatrix} 1 & ax & 0 \\ 1 & bx & 0 \\ 1 & cx & 0 \end{vmatrix} = 0,$$

so $D_1 = 0$.

Simplifying D_2. Split the second column:

$$D_2 = \begin{vmatrix} a & 1 & 1+ax^2 \\ b & 1 & 1+bx^2 \\ c & 1 & 1+cx^2 \end{vmatrix} + \underbrace{\begin{vmatrix} a & ax & 1+ax^2 \\ b & bx & 1+bx^2 \\ c & cx & 1+cx^2 \end{vmatrix}}_{C_2 - xC_1 = 0 \to C_2}.$$

The second determinant is zero, since $C_2 - xC_1 \to C_2$ produces a zero column. Splitting the third column of the first determinant,

$$\begin{vmatrix} a & 1 & 1+ax^2 \\ b & 1 & 1+bx^2 \\ c & 1 & 1+cx^2 \end{vmatrix} = \underbrace{\begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix}}_{0} + \underbrace{\begin{vmatrix} a & 1 & ax^2 \\ b & 1 & bx^2 \\ c & 1 & cx^2 \end{vmatrix}}_{C_3 - x^2C_1 = 0 \to C_3} = 0,$$

so $D_2 = 0$. Therefore

$$\begin{vmatrix} 1+a & 1+ax & 1+ax^2 \\ 1+b & 1+bx & 1+bx^2 \\ 1+c & 1+cx & 1+cx^2 \end{vmatrix} = 0.$$

◀

Example 5.17 (Problem 3)

Let $\mathbf{A} = \begin{bmatrix} x+1 & x+2 & x+3 \\ x & x+3 & x+3 \\ x & x+2 & x+4 \end{bmatrix}$. If $|\mathbf{A}| = 99$, find x.

Solution. Perform $R_1 - R_3 \to R_1$ and $R_2 - R_3 \to R_2$, which do not change the determinant:

$$|\mathbf{A}| = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ x & x+2 & x+4 \end{vmatrix} = (x+4) + (x+2) + x.$$

Setting this equal to 99,

$$x+4+x+2+x = 99 \;\Rightarrow\; 3x+6 = 99 \;\Rightarrow\; x = \frac{93}{3} = 31.$$

◀

Chapter summary

- Order 1: $\det(a_{11}) = a_{11}$. Order 2: $|\mathbf{A}| = a_{11}a_{22} - a_{12}a_{21}$.
- Two equations have a unique solution iff $D = a_1b_2 - a_2b_1 \neq 0$.
- Cofactor $C_{ij} = (-1)^{i+j}|\mathbf{M}_{ij}|$; expand along any row or column.
- Swapping rows negates $|\mathbf{A}|$; scaling a row by k scales it by k.

- Adding a multiple of one row to another leaves $|\mathbf{A}|$ unchanged.
- A zero row, two identical rows, or a row that is a linear combination of others gives $|\mathbf{A}| = 0$.
- $|\mathbf{A}^{\mathsf{T}}| = |\mathbf{A}|$ and $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$; a triangular matrix has determinant equal to the product of its diagonal.
- An orthogonal matrix has $|\mathbf{U}| = \pm 1$.
- $\mathbf{A}$ is invertible if and only if $|\mathbf{A}| \neq 0$.
- $\operatorname{adj}(\mathbf{A})$ is the transpose of the cofactor matrix; $\mathbf{A}\operatorname{adj}(\mathbf{A}) = |\mathbf{A}|\mathbf{I}$, so $\mathbf{A}^{-1} = \operatorname{adj}(\mathbf{A})/|\mathbf{A}|$.
- Cramer's rule: $x_i = |\mathbf{A}_i|/|\mathbf{A}|$ when $|\mathbf{A}| \neq 0$.
- Principal minors have sign $+1$.
- Triangular block matrix: $|\mathbf{M}| = |\mathbf{A}_1||\mathbf{A}_2| \cdots |\mathbf{A}_n|$.
- The volume of the parallelepiped spanned by the columns of $\mathbf{A}$ is $|\det(\mathbf{A})|$.
- $|\mathbf{A}^{\mathsf{H}}| = |\mathbf{A}|^*$; a unitary matrix has $|\det \mathbf{U}| = 1$.

Exercises

5.1 Evaluate $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{vmatrix}$ by expanding along the first row, and check your answer by expanding along the second column.

5.2 Solve $2x + y = 5$, $x + 3y = 10$ using determinants.

5.3 Use the classical adjoint to find the inverse of $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$.

5.4 Show that $\det(k\mathbf{A}) = k^n \det(\mathbf{A})$ for an $n \times n$ matrix $\mathbf{A}$.

5.5 Show that if $\mathbf{A}$ is invertible, then $\det(\mathbf{A}^{-1}) = 1/\det(\mathbf{A})$.

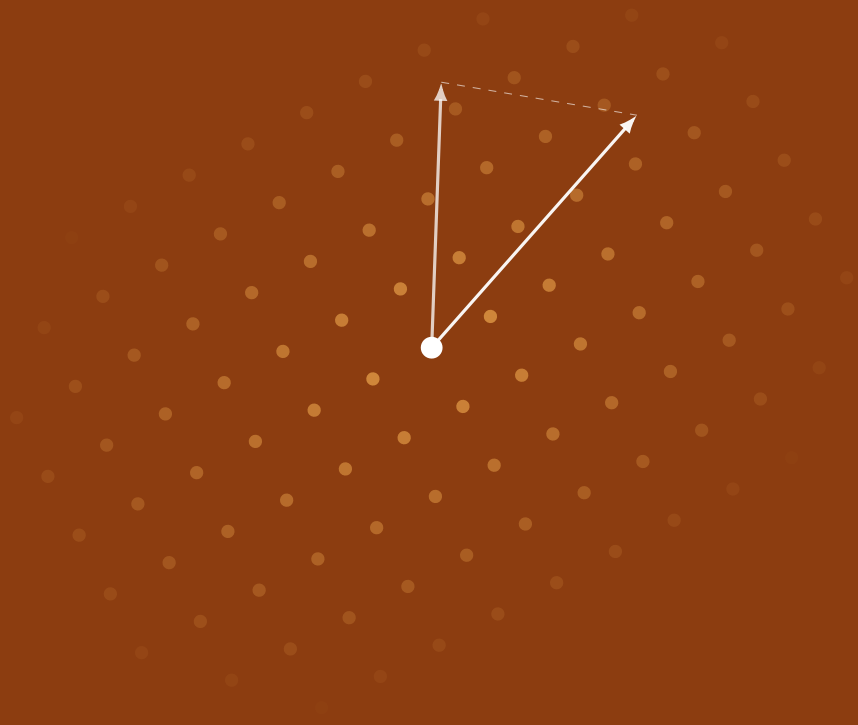
5.6 Find the area of the parallelogram spanned by $\begin{bmatrix} 3 & 1 \end{bmatrix}^{\mathsf{T}}$ and $\begin{bmatrix} 1 & 2 \end{bmatrix}^{\mathsf{T}}$.

5.7 Show that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$.

5.8 Show that the determinant of a skew-symmetric matrix of odd order is zero.

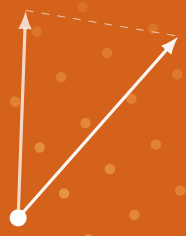
PART IV

Vector and Inner Product Spaces



06

Vector Spaces



Vector spaces, subspaces, span, linear independence, bases, dimension, rank and the four fundamental subspaces

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6.1 Vector spaces

Definition 6.1 (Vector space). A **vector space** $\mathcal{V}$ comprises elements $\bar{\mathbf{u}}, \bar{\mathbf{v}}, \bar{\mathbf{w}}$, termed **vectors**, over an arbitrary **field** $\mathbb{K}$ whose elements $a, b, c \in \mathbb{K}$ are termed **scalars**. In this book $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$.
Let $\mathcal{V}$ be a nonempty set with two operations:

- *Vector addition*, which assigns to any $\bar{\mathbf{u}}, \bar{\mathbf{v}} \in \mathcal{V}$ a sum $\bar{\mathbf{u}} + \bar{\mathbf{v}} \in \mathcal{V}$;
- *Scalar multiplication*, which assigns to any $\bar{\mathbf{u}} \in \mathcal{V}$ and $k \in \mathbb{K}$ a product $k\bar{\mathbf{u}} \in \mathcal{V}$.

Then $\mathcal{V}$ is a vector space over $\mathbb{K}$ if the following axioms hold for any vectors $\bar{\mathbf{u}}, \bar{\mathbf{v}}, \bar{\mathbf{w}} \in \mathcal{V}$ and scalars $a, b, k \in \mathbb{K}$:

1. $(\bar{\mathbf{u}} + \bar{\mathbf{v}}) + \bar{\mathbf{w}} = \bar{\mathbf{u}} + (\bar{\mathbf{v}} + \bar{\mathbf{w}})$.

2. There is a vector in $\mathcal{V}$, denoted $\mathbf{0}$ and termed the **zero vector**, such that $\bar{\mathbf{u}} + \mathbf{0} = \mathbf{0} + \bar{\mathbf{u}} = \bar{\mathbf{u}}$ for any $\bar{\mathbf{u}} \in \mathcal{V}$.
3. For each $\bar{\mathbf{u}} \in \mathcal{V}$ there is a vector in $\mathcal{V}$, denoted $-\bar{\mathbf{u}}$ and called the **negative** of $\bar{\mathbf{u}}$, such that $\bar{\mathbf{u}} + (-\bar{\mathbf{u}}) = (-\bar{\mathbf{u}}) + \bar{\mathbf{u}} = \mathbf{0}$.
4. $\bar{\mathbf{u}} + \bar{\mathbf{v}} = \bar{\mathbf{v}} + \bar{\mathbf{u}}$.
5. $k(\bar{\mathbf{u}} + \bar{\mathbf{v}}) = k\bar{\mathbf{u}} + k\bar{\mathbf{v}}$.
6. $(a + b)\bar{\mathbf{u}} = a\bar{\mathbf{u}} + b\bar{\mathbf{u}}$.
7. $(ab)\bar{\mathbf{u}} = a(b\bar{\mathbf{u}})$.
8. $1\bar{\mathbf{u}} = \bar{\mathbf{u}}$.

Theorem 6.2 (Elementary consequences of the axioms)

Let $\mathcal{V}$ be a vector space over $\mathbb{K}$.

1. *For any scalar $k \in \mathbb{K}$ and $\mathbf{0} \in \mathcal{V}$: $k\mathbf{0} = \mathbf{0}$.*
2. *For $0 \in \mathbb{K}$ and any vector $\bar{\mathbf{u}} \in \mathcal{V}$: $0\bar{\mathbf{u}} = \mathbf{0}$.*
3. *If $k\bar{\mathbf{u}} = \mathbf{0}$, where $k \in \mathbb{K}$ and $\bar{\mathbf{u}} \in \mathcal{V}$, then $k = 0$ or $\bar{\mathbf{u}} = \mathbf{0}$.*
4. *For any $k \in \mathbb{K}$ and any $\bar{\mathbf{u}} \in \mathcal{V}$: $(-k)\bar{\mathbf{u}} = k(-\bar{\mathbf{u}}) = -k\bar{\mathbf{u}}$.*

6.1.1 The space $\mathbb{K}^n$

Let $\mathbb{K}$ be an arbitrary field. The notation $\mathbb{K}^n$ denotes the set of all n-tuples of elements of $\mathbb{K}$: for $a_1, a_2, \ldots, a_n \in \mathbb{K}$,

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{K}^n.$$

$\mathbb{K}^n$ is a vector space over $\mathbb{K}$ with the following operations.

- Vector addition: $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}$.
- Scalar multiplication: $k \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ka_1 \\ ka_2 \\ \vdots \\ ka_n \end{bmatrix}$.
- The negative of a vector: $-\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} -a_1 \\ -a_2 \\ \vdots \\ -a_n \end{bmatrix}$.

- The zero vector is the n-tuple of zeros, $\mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}^\mathsf{T}$.

When $\mathbb{K} = \mathbb{R}$ the space is $\mathbb{R}^n$, a vector space over $\mathbb{R}$; similarly, $\mathbb{C}^n$ is a vector space over $\mathbb{C}$.

6.1.2 Polynomial spaces

Let $\mathcal{P}(t)$ denote the set of all polynomials of the form

$$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_s t^s,$$

and let $\mathcal{P}_n(t)$ denote the set of such polynomials with $s \leq n$, that is, of degree $\leq n$. Then $\mathcal{P}(t)$ and $\mathcal{P}_n(t)$ are vector spaces over $\mathbb{K}$ with the following operations.

- Vector addition: $p(t) + q(t)$ is the usual addition of polynomials.
- Scalar multiplication: $kp(t)$ is the usual product of a scalar k and a polynomial $p(t)$.

The zero polynomial 0 is the zero vector in $\mathcal{P}_n(t)$ or $\mathcal{P}(t)$.

6.1.3 Matrix spaces

The notation $\mathbb{K}^{m\times n}$ denotes the set of all $m \times n$ matrices with entries from $\mathbb{K}$. $\mathbb{K}^{m\times n}$ is a vector space over $\mathbb{K}$ with the usual operations of matrix addition and scalar multiplication of matrices.

6.1.4 Function spaces

Let $\mathcal{X}$ be a non-empty set, let $\mathbb{K}$ be an arbitrary field, and let $\mathcal{F}(\mathcal{X})$ denote the set of all functions from $\mathcal{X}$ into $\mathbb{K}$. Then $\mathcal{F}(\mathcal{X})$ is a vector space over $\mathbb{K}$ with the following operations.

- The sum of two functions $f, g \in \mathcal{F}(\mathcal{X})$ is $(f+g)(x) = f(x) + g(x)$.
- The product of a scalar $k \in \mathbb{K}$ and $f \in \mathcal{F}(\mathcal{X})$ is $(kf)(x) = kf(x)$.
- The zero function maps every element $x \in \mathcal{X}$ to $0 \in \mathbb{K}$: $0(x) = 0$.
- The negative of a function is defined by $(-f)(x) = -f(x)$.

6.2 Linear combinations

Definition 6.3 (Linear combination). Let $\mathcal{V}$ be a vector space over a field $\mathbb{K}$. A vector $\bar{\mathbf{v}} \in \mathcal{V}$ is a **linear combination** of $\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \ldots, \bar{\mathbf{u}}_m$ if there exist scalars $a_1, a_2, \ldots, a_m \in \mathbb{K}$ such that

$$\bar{\mathbf{v}} = a_1\bar{\mathbf{u}}_1 + a_2\bar{\mathbf{u}}_2 + \cdots + a_m\bar{\mathbf{u}}_m. \tag{6.1}$$

Equivalently, $\bar{\mathbf{v}}$ is a linear combination of $\bar{\mathbf{u}}_1, \ldots, \bar{\mathbf{u}}_m$ if the vector equation

$$\bar{\mathbf{v}} = x_1\bar{\mathbf{u}}_1 + x_2\bar{\mathbf{u}}_2 + \cdots + x_m\bar{\mathbf{u}}_m$$

has a solution in scalars $x_1, x_2, \ldots, x_m$.

Example 6.1 (Linear combination in $\mathbb{R}^3$)

Express $\bar{\mathbf{v}} = \begin{bmatrix} 1 & 0 & -6 \end{bmatrix}^T$ as a linear combination of

$$\bar{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \qquad \bar{\mathbf{u}}_2 = \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}, \qquad \bar{\mathbf{u}}_3 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}.$$

Solution. We seek scalars x, y, z such that

$$\begin{bmatrix} 1 \\ 0 \\ -6 \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix} + z \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}.$$

This leads to the system

$$x + 2y + z = 1, \qquad 2x + 5y + 3z = 0, \qquad x + 4y + 4z = -6.$$

Reducing it to echelon form gives

$$x + 2y + z = 1, \qquad y + z = -2, \qquad z = -3,$$

so $x = 2, y = 1, z = -3$, and

$$\bar{\mathbf{v}} = 2\bar{\mathbf{u}}_1 + \bar{\mathbf{u}}_2 - 3\bar{\mathbf{u}}_3.$$

◀

In general, expressing a given vector $\bar{\mathbf{v}}$ in $\mathbb{K}^n$ as a linear combination of vectors $\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \ldots, \bar{\mathbf{u}}_m$ in $\mathbb{K}^n$ is equivalent to solving the system

$$\begin{bmatrix} | & | & & | \\ \bar{\mathbf{u}}_1 & \bar{\mathbf{u}}_2 & \cdots & \bar{\mathbf{u}}_m \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \qquad \text{that is,} \qquad \mathbf{U}\bar{\mathbf{x}} = \bar{\mathbf{v}}.$$

As seen in Chapter 4, such a system may have a unique solution, many solutions, or no solution. No solution means that $\bar{\mathbf{v}}$ cannot be written as a linear combination of the $\bar{\mathbf{u}}_i$.

Example 6.2 (Linear combination of polynomials)

Express $v(t) = 4t^2 + 7t + 4$ as a linear combination of

$$p_1(t) = t^2 + t + 1, \qquad p_2(t) = 2t^2 + 3t + 1, \qquad p_3(t) = t^2 + 2t + 3.$$

Solution. We seek scalars x, y, z such that

$$4t^2 + 7t + 4 = x(t^2 + t + 1) + y(2t^2 + 3t + 1) + z(t^2 + 2t + 3),$$

which can be rewritten as

$$4t^2 + 7t + 4 = t^2(x + 2y + z) + t(x + 3y + 2z) + (x + y + 3z).$$

Equating coefficients and reducing to echelon form,

$$x + 2y + z = 4, \qquad y + z = 3, \qquad 3z = 3.$$

The system has the solution $x = -1, y = 2, z = 1$, so

$$v(t) = -p_1(t) + 2p_2(t) + p_3(t).$$

◀

6.3 Spanning sets

Definition 6.4 (Spanning set). Let $\mathcal{V}$ be a vector space over the field $\mathbb{K}$. Vectors $\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \ldots, \bar{\mathbf{u}}_m$ in $\mathcal{V}$ **span** $\mathcal{V}$, or form a **spanning set** of $\mathcal{V}$, if every vector $\bar{\mathbf{v}} \in \mathcal{V}$ is a linear combination of them; that is, there exist scalars $a_1, a_2, \ldots, a_m$ such that

$$\bar{\mathbf{v}} = a_1\bar{\mathbf{u}}_1 + a_2\bar{\mathbf{u}}_2 + \cdots + a_m\bar{\mathbf{u}}_m.$$

Example 6.3 (Spanning sets of $\mathbb{R}^3$)

Solution. *First spanning set.* The vectors

$$\bar{\mathbf{e}}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \qquad \bar{\mathbf{e}}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \qquad \bar{\mathbf{e}}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

span $\mathbb{R}^3$: any $\bar{\mathbf{v}} = \begin{bmatrix}a & b & c\end{bmatrix}^\mathsf{T}$ can be written

$$\bar{\mathbf{v}} = a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix} + c\begin{bmatrix}0\\0\\1\end{bmatrix} = a\bar{\mathbf{e}}_1 + b\bar{\mathbf{e}}_2 + c\bar{\mathbf{e}}_3.$$

Second spanning set. The vectors

$$\bar{\mathbf{w}}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \qquad \bar{\mathbf{w}}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \qquad \bar{\mathbf{w}}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

also span $\mathbb{R}^3$: for any $\bar{\mathbf{v}} = \begin{bmatrix}a & b & c\end{bmatrix}^\mathsf{T}$,

$$\bar{\mathbf{v}} = a\begin{bmatrix}1\\1\\1\end{bmatrix} + (b-a)\begin{bmatrix}0\\1\\1\end{bmatrix} + (c-b)\begin{bmatrix}0\\0\\1\end{bmatrix} = a\bar{\mathbf{w}}_1 + (b-a)\bar{\mathbf{w}}_2 + (c-b)\bar{\mathbf{w}}_3.$$

Thus a spanning set is *not* unique. ◀

Example 6.4 (Spanning sets of polynomials and matrices)

Solution. Consider $\mathcal{V} = \mathcal{P}_n(t)$, the polynomials of degree $\leq n$. Any polynomial in $\mathcal{P}_n(t)$

can be expressed as a linear combination of

$$1,\ t,\ t^2,\ \ldots,\ t^n,$$

so these form a spanning set. Any polynomial in $\mathcal{P}_n(t)$ can also be expressed as a linear combination of

$$1,\ t-c,\ (t-c)^2,\ \ldots,\ (t-c)^n,$$

which is another spanning set.
The space $\mathbb{R}^{2\times 2}$ of all 2×2 matrices is spanned by

$$\mathbf{E}_{11}=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\quad \mathbf{E}_{12}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\quad \mathbf{E}_{21}=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix},\quad \mathbf{E}_{22}=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}.$$

◀

6.4 Subspaces

Definition 6.5 (Subspace). Let $\mathcal{V}$ be a vector space over a field $\mathbb{K}$ and let $\mathcal{W}$ be a subset of $\mathcal{V}$. Then $\mathcal{W}$ is a **subspace** of $\mathcal{V}$ if $\mathcal{W}$ is itself a vector space over $\mathbb{K}$ with respect to the operations of vector addition and scalar multiplication on $\mathcal{V}$.

Theorem 6.6 (Subspace test)

A subset $\mathcal{W}$ of a vector space $\mathcal{V}$ is a subspace of $\mathcal{V}$ if the following two conditions hold:

1. *the zero vector $\mathbf{0}$ belongs to $\mathcal{W}$;*
2. *for every $\bar{\mathbf{u}},\bar{\mathbf{v}}\in\mathcal{W}$ and $a,b\in\mathbb{K}$, the linear combination $a\bar{\mathbf{u}}+b\bar{\mathbf{v}}\in\mathcal{W}$.*

Definition 6.7 (Trivial subspaces). Any vector space $\mathcal{V}$ automatically contains two subspaces: the set $\{\mathbf{0}\}$ consisting of the zero vector alone, and the whole space $\mathcal{V}$ itself. These are called the **trivial subspaces** of $\mathcal{V}$.

Example 6.5 (A line in $\mathbb{R}^3$)

Let $\mathcal{U}$ consist of all vectors in $\mathcal{V}=\mathbb{R}^3$ whose entries are equal:

$$\mathcal{U}=\left\{\begin{bmatrix}a & b & c\end{bmatrix}^{\mathsf{T}} : a=b=c\right\}.$$

Show that $\mathcal{U}$ is a subspace of $\mathbb{R}^3$.

Solution. Note that $\mathbf{0}\in\mathcal{U}$. For $\bar{\mathbf{u}}=\begin{bmatrix}a & a & a\end{bmatrix}^{\mathsf{T}}$ and $\bar{\mathbf{v}}=\begin{bmatrix}b & b & b\end{bmatrix}^{\mathsf{T}}$ in $\mathcal{U}$,

$$\alpha\bar{\mathbf{u}}+\beta\bar{\mathbf{v}}=\begin{bmatrix}\alpha a+\beta b\\ \alpha a+\beta b\\ \alpha a+\beta b\end{bmatrix}\in\mathcal{U}.$$

Thus $\mathcal{U}$ is a subspace of $\mathbb{R}^3$. Geometrically, $\mathcal{U}$ is the line through the origin O and the point $\begin{bmatrix}1 & 1 & 1\end{bmatrix}^\mathsf{T}$ (Figure 6.1). ◂

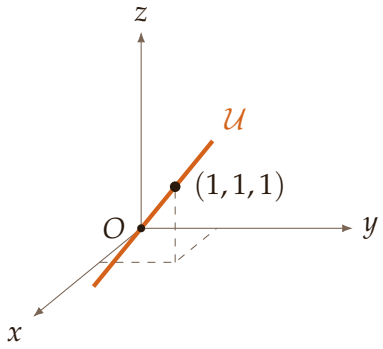


Figure 6.1 The subspace $\mathcal{U}$ of vectors with equal entries is the line through the origin and $(1, 1, 1)$.

Example 6.6 (Subspaces of matrices)

Let $\mathcal{V} = \mathbb{R}^{n\times n}$, let $\mathcal{W}_1$ be the subset of all (upper) triangular matrices, and let $\mathcal{W}_2$ be the subset of all symmetric matrices.

Solution. $\mathcal{W}_1$ is a subspace of $\mathcal{V}$: it contains the zero matrix $\mathbf{0}$ and is closed under matrix addition and scalar multiplication, since the sum and scalar multiple of triangular matrices are also triangular. Similarly, $\mathcal{W}_2$ is a subspace of $\mathcal{V}$. ◂

Example 6.7 (Subspace of polynomials)

Let $\mathcal{V} = \mathcal{P}(t)$, and let $\mathcal{Q}(t)$ be the collection of polynomials with only even powers of t. For example,

$$p_1(t) = 2 - 6t^2 + 8t^4, \qquad p_2(t) = -5 + t^4 + 4t^8 - 2t^{10}$$

are polynomials in $\mathcal{Q}(t)$. Then $\mathcal{Q}(t)$ is a subspace of $\mathcal{P}(t)$.

Theorem 6.8 (Intersection of subspaces)

If $\mathcal{U}$ and $\mathcal{W}$ are subspaces of a vector space $\mathcal{V}$, then the intersection $\mathcal{U} \cap \mathcal{W}$ is also a subspace of $\mathcal{V}$.

Proof. Clearly $\mathbf{0} \in \mathcal{U}$ and $\mathbf{0} \in \mathcal{W}$, because $\mathcal{U}$ and $\mathcal{W}$ are subspaces; whence $\mathbf{0} \in \mathcal{U} \cap \mathcal{W}$. Suppose $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ belong to $\mathcal{U} \cap \mathcal{W}$. Then $\bar{\mathbf{u}}, \bar{\mathbf{v}} \in \mathcal{U}$ and $\bar{\mathbf{u}}, \bar{\mathbf{v}} \in \mathcal{W}$. Because $\mathcal{U}$ and $\mathcal{W}$ are subspaces, for any scalars $a, b \in \mathbb{K}$,

$$a\bar{\mathbf{u}} + b\bar{\mathbf{v}} \in \mathcal{U}, \qquad a\bar{\mathbf{u}} + b\bar{\mathbf{v}} \in \mathcal{W} \;\Rightarrow\; a\bar{\mathbf{u}} + b\bar{\mathbf{v}} \in \mathcal{U} \cap \mathcal{W}.$$

Hence $\mathcal{U} \cap \mathcal{W}$ is a subspace. ■

Theorem 6.9 (Solutions of a homogeneous system)

*The solution set $\mathcal{W}$ of a homogeneous system $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$ in n unknowns is a subspace of $\mathbb{K}^n$, called the **null space** of $\mathbf{A}$.*

Proof. Because $\mathbf{A0} = \mathbf{0}$, the zero vector belongs to $\mathcal{W}$. Suppose $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ belong to $\mathcal{W}$, so that they are solutions of $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$:

$$\mathbf{A}\bar{\mathbf{u}} = \mathbf{0}, \qquad \mathbf{A}\bar{\mathbf{v}} = \mathbf{0}.$$

Then for any scalars a and b,

$$\mathbf{A}(a\bar{\mathbf{u}} + b\bar{\mathbf{v}}) = a\mathbf{A}\bar{\mathbf{u}} + b\mathbf{A}\bar{\mathbf{v}} = a\mathbf{0} + b\mathbf{0} = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Thus $a\bar{\mathbf{u}} + b\bar{\mathbf{v}}$ belongs to $\mathcal{W}$, and $\mathcal{W}$ is a subspace of $\mathbb{K}^n$. ■

6.5 Linear spans

Definition 6.10 (Linear span). Suppose $\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \ldots, \bar{\mathbf{u}}_m$ are any vectors in a vector space $\mathcal{V}$. Recall that any vector of the form $a_1\bar{\mathbf{u}}_1 + a_2\bar{\mathbf{u}}_2 + \cdots + a_m\bar{\mathbf{u}}_m$, where the a_i are scalars, is a linear combination of the $\bar{\mathbf{u}}_i$. The collection of all such linear combinations is the **span** of the vectors, denoted

$$\operatorname{span}(\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \ldots, \bar{\mathbf{u}}_m) \quad \text{or} \quad \operatorname{span}(\bar{\mathbf{u}}_i).$$

Theorem 6.11 (A span is a subspace)

$\operatorname{span}(\bar{\mathbf{u}}_i)$ *is a subspace of* $\mathcal{V}$.

Proof. The zero vector belongs to $\operatorname{span}(\bar{\mathbf{u}}_i)$, because

$$\mathbf{0} = 0\bar{\mathbf{u}}_1 + 0\bar{\mathbf{u}}_2 + \cdots + 0\bar{\mathbf{u}}_m.$$

Suppose $\bar{\mathbf{v}}$ and $\bar{\mathbf{w}}$ belong to $\operatorname{span}(\bar{\mathbf{u}}_i)$, say

$$\bar{\mathbf{v}} = a_1\bar{\mathbf{u}}_1 + a_2\bar{\mathbf{u}}_2 + \cdots + a_m\bar{\mathbf{u}}_m, \qquad \bar{\mathbf{w}} = b_1\bar{\mathbf{u}}_1 + b_2\bar{\mathbf{u}}_2 + \cdots + b_m\bar{\mathbf{u}}_m.$$

Then

$$\alpha\bar{\mathbf{v}} + \beta\bar{\mathbf{w}} = (\alpha a_1 + \beta b_1)\bar{\mathbf{u}}_1 + \cdots + (\alpha a_m + \beta b_m)\bar{\mathbf{u}}_m \in \operatorname{span}(\bar{\mathbf{u}}_i).$$

Thus $\operatorname{span}(\bar{\mathbf{u}}_i)$ is a subspace of $\mathcal{V}$. ■

Example 6.8 (Lines and planes as spans)

Solution. Let $\bar{\mathbf{u}}$ be any nonzero vector in $\mathbb{R}^3$. Then $\operatorname{span}(\bar{\mathbf{u}})$ consists of all scalar multiples of $\bar{\mathbf{u}}$. Geometrically, it is the line through the origin O and the endpoint of $\bar{\mathbf{u}}$ (Figure 6.2, left).
Let $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ be vectors in $\mathbb{R}^3$ that are not multiples of each other. Then $\operatorname{span}(\bar{\mathbf{u}}, \bar{\mathbf{v}})$ is the plane through the origin O and the endpoints of $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ (Figure 6.2, right).
Finally, for the standard vectors $\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3$ of Example 6.3, $\operatorname{span}(\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3) = \mathbb{R}^3$. ◂

6.6 Row and column spaces

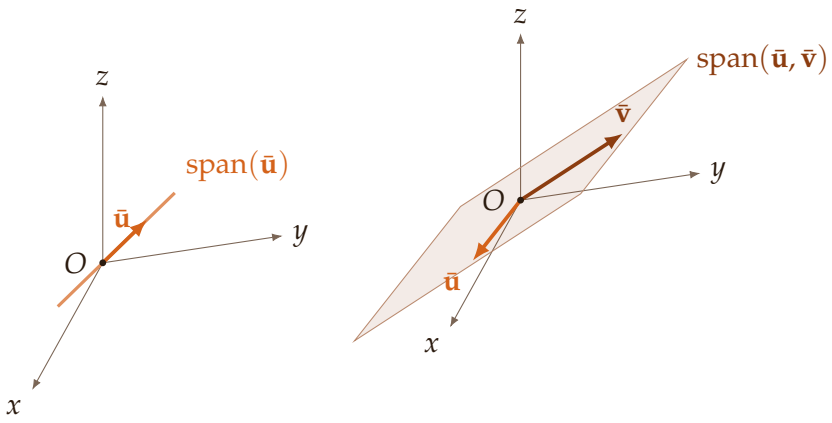


Figure 6.2 Left: $\text{span}(\bar{\mathbf{u}})$ is the line through the origin along $\bar{\mathbf{u}}$. Right: $\text{span}(\bar{\mathbf{u}}, \bar{\mathbf{v}})$ is the plane through the origin containing $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$.

Definition 6.12 (Column space and row space). Let $\mathbf{A} = [a_{ij}]$ be an arbitrary $m \times n$ matrix with columns $\bar{\mathbf{a}}_1, \bar{\mathbf{a}}_2, \ldots, \bar{\mathbf{a}}_n$. The **column space** of $\mathbf{A}$ is

$$\text{colsp}(\mathbf{A}) = \text{span}(\bar{\mathbf{a}}_1, \bar{\mathbf{a}}_2, \ldots, \bar{\mathbf{a}}_n).$$

Similarly, the **row space** $\text{rowsp}(\mathbf{A})$ is the span of the rows of $\mathbf{A}$. Note that

$$\text{rowsp}(\mathbf{A}) = \text{colsp}(\mathbf{A}^{\mathsf{T}}). \tag{6.2}$$

Recall that $\mathbf{A}$ and $\mathbf{B}$ are **row equivalent**, written $\mathbf{A} \sim \mathbf{B}$, if $\mathbf{B}$ can be obtained from $\mathbf{A}$ by a sequence of elementary row operations

$$\bar{\mathbf{r}}_i \leftrightarrow \bar{\mathbf{r}}_j, \qquad k\bar{\mathbf{r}}_i \to \bar{\mathbf{r}}_i, \qquad \bar{\mathbf{r}}_j + k\bar{\mathbf{r}}_i \to \bar{\mathbf{r}}_j.$$

Theorem 6.13 (Row-equivalent matrices have the same row space)

If $\mathbf{A} \sim \mathbf{B}$, *then* $\text{rowsp}(\mathbf{A}) = \text{rowsp}(\mathbf{B})$.

Proof. Suppose $\mathbf{M}$ is obtained from $\mathbf{A}$ by one elementary row operation. Each row of $\mathbf{M}$ is a row of $\mathbf{A}$ or a linear combination of rows of $\mathbf{A}$, so the row space of $\mathbf{M}$ is contained in the row space of $\mathbf{A}$. On the other hand, applying the inverse elementary row operation to $\mathbf{M}$ gives $\mathbf{A}$, so the row space of $\mathbf{A}$ is contained in the row space of $\mathbf{M}$. Accordingly, $\mathbf{A}$ and $\mathbf{M}$ have the same row space. Repeating the argument for each operation in the sequence proves the theorem. ■

Theorem 6.14 (Row canonical matrices)

Suppose $\mathbf{A}$ *and* $\mathbf{B}$ *are row canonical matrices. Then* $\mathbf{A}$ *and* $\mathbf{B}$ *have the same row space if and only if they have the same nonzero rows.*

Example 6.9 (Equal spans via row canonical form)

Let $\mathcal{U} = \operatorname{span}(\bar{\mathbf{u}}_i)$ and $\mathcal{W} = \operatorname{span}(\bar{\mathbf{w}}_i)$, where

$$\bar{\mathbf{u}}_1^{\mathsf{T}} = \begin{bmatrix} 1 & -3 & 2 & 12 \end{bmatrix}, \qquad \bar{\mathbf{w}}_1^{\mathsf{T}} = \begin{bmatrix} 1 & -3 & -1 & -3 \end{bmatrix},$$
$$\bar{\mathbf{u}}_2^{\mathsf{T}} = \begin{bmatrix} 2 & -6 & -1 & -1 \end{bmatrix}, \qquad \bar{\mathbf{w}}_2^{\mathsf{T}} = \begin{bmatrix} 2 & -6 & 3 & 19 \end{bmatrix},$$
$$\bar{\mathbf{u}}_3^{\mathsf{T}} = \begin{bmatrix} 3 & -9 & 1 & 11 \end{bmatrix}.$$

Show that $\mathcal{U} = \mathcal{W}$.

Solution. Form the matrix $\mathbf{A}$ with rows $\bar{\mathbf{u}}_i^{\mathsf{T}}$ and reduce it:

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 2 & 12 \\ 2 & -6 & -1 & -1 \\ 3 & -9 & 1 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 12 \\ 0 & 0 & -5 & -25 \\ 0 & 0 & -5 & -25 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Form the matrix $\mathbf{B}$ with rows $\bar{\mathbf{w}}_i^{\mathsf{T}}$ and reduce it:

$$\mathbf{B} = \begin{bmatrix} 1 & -3 & -1 & -3 \\ 2 & -6 & 3 & 19 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -1 & -3 \\ 0 & 0 & 5 & 25 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}.$$

The row canonical forms of $\mathbf{A}$ and $\mathbf{B}$ have the same nonzero rows,

$$\begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix},$$

so the row spaces of $\mathbf{A}$ and $\mathbf{B}$ are identical. Therefore

$$\mathcal{U} = \operatorname{span}(\bar{\mathbf{u}}_i) = \mathcal{W} = \operatorname{span}(\bar{\mathbf{w}}_i).$$

◀

6.7 Linear dependence and independence

Definition 6.15 (Linear dependence and independence). Let $\mathcal{V}$ be a vector space over a field $\mathbb{K}$. Vectors $\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \ldots, \bar{\mathbf{v}}_n \in \mathcal{V}$ are **linearly dependent** if there exist scalars $a_1, a_2, \ldots, a_n$, not all zero, such that

$$a_1\bar{\mathbf{v}}_1 + a_2\bar{\mathbf{v}}_2 + \cdots + a_n\bar{\mathbf{v}}_n = \mathbf{0}.$$

Otherwise they are **linearly independent**; in other words,

$$a_1\bar{\mathbf{v}}_1 + a_2\bar{\mathbf{v}}_2 + \cdots + a_n\bar{\mathbf{v}}_n = \mathbf{0} \iff a_1 = a_2 = \cdots = a_n = 0. \tag{6.3}$$

Equivalently, $\bar{\mathbf{v}}_1, \dots, \bar{\mathbf{v}}_n$ are linearly dependent if the equation

$$\begin{bmatrix} | & | & & | \\ \bar{\mathbf{v}}_1 & \bar{\mathbf{v}}_2 & \cdots & \bar{\mathbf{v}}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \mathbf{0}, \qquad \text{that is,} \qquad \mathbf{V}\bar{\mathbf{a}} = \mathbf{0},$$

has a nonzero solution $\bar{\mathbf{a}}$.

Theorem 6.16 (Properties of dependence and independence)

1. *If $\mathbf{0}$ is one of the vectors $\bar{\mathbf{v}}_1, \dots, \bar{\mathbf{v}}_n$, the vectors are linearly dependent.*
2. *If $\bar{\mathbf{v}} \neq \mathbf{0}$, then $\bar{\mathbf{v}}$ by itself is linearly independent.*
3. *If one of the vectors $\bar{\mathbf{v}}_1, \dots, \bar{\mathbf{v}}_n$ is a scalar multiple of another, the vectors are linearly dependent.*
4. *Two vectors are linearly independent if and only if neither is a multiple of the other.*
5. *If a set S of vectors is linearly independent, then any subset of S is linearly independent.*

Proof of (1)–(3). (1) Suppose $\bar{\mathbf{v}}_1 = \mathbf{0}$. Then $1\bar{\mathbf{v}}_1 + 0\bar{\mathbf{v}}_2 + \cdots + 0\bar{\mathbf{v}}_n = \mathbf{0}$ with a nonzero coefficient. (2) By Theorem 6.2(3), $k\bar{\mathbf{v}} = \mathbf{0}$ with $\bar{\mathbf{v}} \neq \mathbf{0}$ forces $k = 0$. (3) Without loss of generality let $\bar{\mathbf{v}}_1 = k\bar{\mathbf{v}}_2$. Then $1\bar{\mathbf{v}}_1 - k\bar{\mathbf{v}}_2 + 0\bar{\mathbf{v}}_3 + \cdots + 0\bar{\mathbf{v}}_n = \mathbf{0}$. ■

Example 6.10 (Testing independence)

Determine whether

$$\bar{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \qquad \bar{\mathbf{u}}_2 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \qquad \bar{\mathbf{u}}_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

are linearly independent or dependent.

Solution. Form the equation

$$x\bar{\mathbf{u}}_1 + y\bar{\mathbf{u}}_2 + z\bar{\mathbf{u}}_3 = x\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + z\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \mathbf{0}.$$

The equivalent system

$$x + 2y + z = 0, \qquad -x - y + z = 0, \qquad 2x + 3y + 4z = 0$$

reduces to

$$x + 2y + z = 0, \qquad y + 2z = 0, \qquad 4z = 0.$$

Back-substitution yields $x = y = z = 0$. Therefore $\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \bar{\mathbf{u}}_3$ are linearly independent. ◂

Theorem 6.17 (Dependence and linear combinations)

Vectors $\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \dots, \bar{\mathbf{v}}_n$ are linearly dependent if and only if one of them is a linear combination of the others.

Proof. Suppose $\bar{\mathbf{v}}_i$ is a linear combination of the others,

$$\bar{\mathbf{v}}_i = a_1\bar{\mathbf{v}}_1 + \cdots + a_{i-1}\bar{\mathbf{v}}_{i-1} + a_{i+1}\bar{\mathbf{v}}_{i+1} + \cdots + a_n\bar{\mathbf{v}}_n.$$

It follows that

$$\bar{\mathbf{v}}_i - a_1\bar{\mathbf{v}}_1 - \cdots - a_{i-1}\bar{\mathbf{v}}_{i-1} - a_{i+1}\bar{\mathbf{v}}_{i+1} - \cdots - a_n\bar{\mathbf{v}}_n = \mathbf{0},$$

with coefficient 1 on $\bar{\mathbf{v}}_i$, so the vectors are linearly dependent. Conversely, suppose the vectors are linearly dependent, $a_1\bar{\mathbf{v}}_1 + a_2\bar{\mathbf{v}}_2 + \cdots + a_n\bar{\mathbf{v}}_n = \mathbf{0}$, with some $a_i \neq 0$. Then

$$\bar{\mathbf{v}}_i = -\frac{a_1}{a_i}\bar{\mathbf{v}}_1 - \cdots - \frac{a_{i-1}}{a_i}\bar{\mathbf{v}}_{i-1} - \frac{a_{i+1}}{a_i}\bar{\mathbf{v}}_{i+1} - \cdots - \frac{a_n}{a_i}\bar{\mathbf{v}}_n,$$

which completes the proof. ∎

6.8 Basis and dimension

Definition 6.18 (Basis). A set $S = \{\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \ldots, \bar{\mathbf{u}}_m\}$ of vectors is a **basis** of $\mathcal{V}$ if it is linearly independent and spans $\mathcal{V}$.

Theorem 6.19 (Invariance of the number of basis vectors)

Let $\mathcal{V}$ be a vector space such that one basis has m elements and another basis has n elements. Then $m = n$.

Definition 6.20 (Dimension). By Theorem 6.19, all bases of $\mathcal{V}$ have the same number of elements n. This unique number is the **dimension** of $\mathcal{V}$, written $\dim \mathcal{V} = n$; alternatively, $\mathcal{V}$ is said to be ***n*-dimensional**.

Example 6.11 (The standard basis of $\mathbb{K}^n$)

Solution. The vectors

$$\bar{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \bar{\mathbf{e}}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \bar{\mathbf{e}}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

are linearly independent. Furthermore, any vector in $\mathbb{K}^n$ can be written as a linear combination of them:

$$\bar{\mathbf{u}} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = u_1\bar{\mathbf{e}}_1 + u_2\bar{\mathbf{e}}_2 + \cdots + u_n\bar{\mathbf{e}}_n.$$

Thus $\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \ldots, \bar{\mathbf{e}}_n$ form a basis of $\mathbb{R}^n$, and its dimension is n. ◂

Example 6.12 (Basis of polynomials)

The set $S = \{1, t, t^2, \ldots, t^n\}$ of $n+1$ polynomials is a basis of the vector space $\mathcal{P}_n(t)$ of all polynomials of degree $\leq n$, so $\dim \mathcal{P}_n(t) = n+1$.

Theorem 6.21 (Properties of bases)

Let $\mathcal{V}$ be a vector space of finite dimension n. Then:

1. *any $n+1$ or more vectors in $\mathcal{V}$ are linearly dependent;*
2. *any linearly independent set $S = \{\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \ldots, \bar{\mathbf{u}}_n\}$ with n elements is a basis of $\mathcal{V}$;*
3. *any spanning set $T = \{\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \ldots, \bar{\mathbf{v}}_n\}$ of $\mathcal{V}$ with n elements is a basis of $\mathcal{V}$.*

Theorem 6.22 (Dimension of a subspace)

Let $\mathcal{W}$ be a subspace of an n-dimensional vector space $\mathcal{V}$. Then $\dim \mathcal{W} \leq n$. In particular, if $\dim \mathcal{W} = n$, then $\mathcal{W} = \mathcal{V}$.

6.9 Rank of a matrix

Let $\mathbf{A}$ be any $m \times n$ matrix over a field $\mathbb{K}$. Its rows may be viewed as vectors in $\mathbb{K}^n$, and its **row space** $\operatorname{rowsp}(\mathbf{A})$ is the subspace of $\mathbb{K}^n$ they span. Its columns may be viewed as vectors in $\mathbb{K}^m$, and its **column space** $\operatorname{colsp}(\mathbf{A})$ is the subspace of $\mathbb{K}^m$ they span.

Definition 6.23 (Rank). The **rank** of $\mathbf{A}$, written $\operatorname{rank}(\mathbf{A})$, is the maximum number of linearly independent rows of $\mathbf{A}$, or equivalently, the dimension of the row space of $\mathbf{A}$.

Theorem 6.24 (Row rank equals column rank)

The maximum number of linearly independent rows of any matrix $\mathbf{A}$ equals the maximum number of linearly independent columns of $\mathbf{A}$. Therefore the row space and the column space of $\mathbf{A}$ have the same dimension:

$$\dim \operatorname{rowsp}(\mathbf{A}) = \dim \operatorname{colsp}(\mathbf{A}) = \operatorname{rank}(\mathbf{A}). \tag{6.4}$$

Example 6.13 (Rank and bases of row and column spaces)

Consider the matrix $\mathbf{A}$ below and its echelon form $\mathbf{B}$:

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 & 2 & 0 & 1 \\ 2 & -1 & 5 & 3 & 2 & 5 \\ 3 & -1 & 8 & 5 & 5 & 7 \\ 1 & 2 & 5 & 1 & 8 & 6 \\ 2 & -1 & 5 & 4 & 3 & 3 \end{bmatrix} \sim \mathbf{B} = \begin{bmatrix} 1 & -1 & 2 & 2 & 0 & 1 \\ 0 & 1 & 1 & -1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find the rank of $\mathbf{A}$ and bases of its row space and column space.

Solution. *Rank.* $\mathbf{A}$ and $\mathbf{B}$ are row equivalent, so they have the same row space. Counting pivots, $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{B}) = 3$.
Row space. $\mathbf{B}$ is in echelon form, so its nonzero rows are linearly independent and form a basis of the row space of $\mathbf{B}$. Since $\mathbf{A}$ and $\mathbf{B}$ are row equivalent, they also form a basis of the row space of $\mathbf{A}$:

$$\begin{bmatrix}1 & -1 & 2 & 2 & 0 & 1\end{bmatrix}, \quad \begin{bmatrix}0 & 1 & 1 & -1 & 2 & 3\end{bmatrix}, \quad \begin{bmatrix}0 & 0 & 0 & 1 & 1 & -2\end{bmatrix}.$$

Column space. The columns $\bar{\mathbf{c}}_1, \bar{\mathbf{c}}_2, \bar{\mathbf{c}}_4$ of $\mathbf{B}$ associated with the pivots are linearly independent. Therefore columns $\bar{\mathbf{c}}_1, \bar{\mathbf{c}}_2, \bar{\mathbf{c}}_4$ of $\mathbf{A}$ are also linearly independent and form a basis of $\operatorname{colsp}(\mathbf{A})$:

$$\begin{bmatrix}1\\2\\3\\1\\2\end{bmatrix}, \quad \begin{bmatrix}-1\\-1\\-1\\2\\-1\end{bmatrix}, \quad \begin{bmatrix}2\\3\\5\\1\\4\end{bmatrix}.$$

◀

Remark. Row operations preserve the *linear relations among the columns*, but not the column space itself. That is why the column-space basis is taken from the pivot columns of the original $\mathbf{A}$, not of $\mathbf{B}$.

6.10 Null space and nullity

Consider again a homogeneous system $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$ of linear equations over $\mathbb{K}$ in n unknowns. As seen in Theorem 6.9, its solution set is a subspace of $\mathbb{K}^n$, the **null space** of $\mathbf{A}$.

Theorem 6.25 (Dimension of the null space)

The dimension of the solution space $\mathcal{W}$ of a homogeneous system $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$ is $n - r$, where n is the number of unknowns and r is the rank of the coefficient matrix $\mathbf{A}$.

The dimension of the null space of a matrix is called its **nullity**. We therefore have the **rank–nullity** result

$$\text{rank} + \text{nullity} = r + (n - r) = n = \text{number of columns of } \mathbf{A}. \tag{6.5}$$

Example 6.14 (A basis of the null space)

Since the null space of a matrix is a vector space, one can find a basis for it. Find a basis of the null space of $\mathbf{A} = \begin{bmatrix}1 & -3 & 0 & 2\\0 & 0 & 1 & -4\end{bmatrix}$.

Solution. Set $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$:

$$\begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{0}.$$

Solving the equations,

$$\begin{aligned} x_1 - 3x_2 + 0x_3 + 2x_4 = 0 \;&\Rightarrow\; x_1 = 3x_2 - 2x_4, \\ x_3 - 4x_4 = 0 \;&\Rightarrow\; x_3 = 4x_4. \end{aligned}$$

Therefore any vector of the null space can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_2 - 2x_4 \\ x_2 \\ 4x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 4 \\ 1 \end{bmatrix},$$

and a basis of the null space is

$$\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ 0 \\ 4 \\ 1 \end{bmatrix}.$$

Here $n = 4$ and $r = 2$, so the nullity is $n - r = 2$, as (6.5) predicts. ◂

6.11 The four fundamental subspaces

Theorem 6.26 (Four fundamental subspaces)

Every $m \times n$ matrix $\mathbf{A}$ of rank r has four fundamental subspaces:

1. *the row space of $\mathbf{A}$, of dimension r, in $\mathbb{K}^n$;*
2. *the column space of $\mathbf{A}$, of dimension r, in $\mathbb{K}^m$;*
3. *the null space of $\mathbf{A}$, of dimension $n - r$, in $\mathbb{K}^n$;*
4. *the null space of $\mathbf{A}^\mathsf{T}$, of dimension $m - r$, in $\mathbb{K}^m$.*

They satisfy:

- *the row space of $\mathbf{A}$ is orthogonal to the null space of $\mathbf{A}$;*
- *the column space of $\mathbf{A}$ is orthogonal to the null space of $\mathbf{A}^\mathsf{T}$.*

AIML & 6G Connection

The rank of a MIMO channel matrix $\mathbf{H}$ is the number of independent data streams it can carry: its column space holds every received signal the transmitter can produce. In

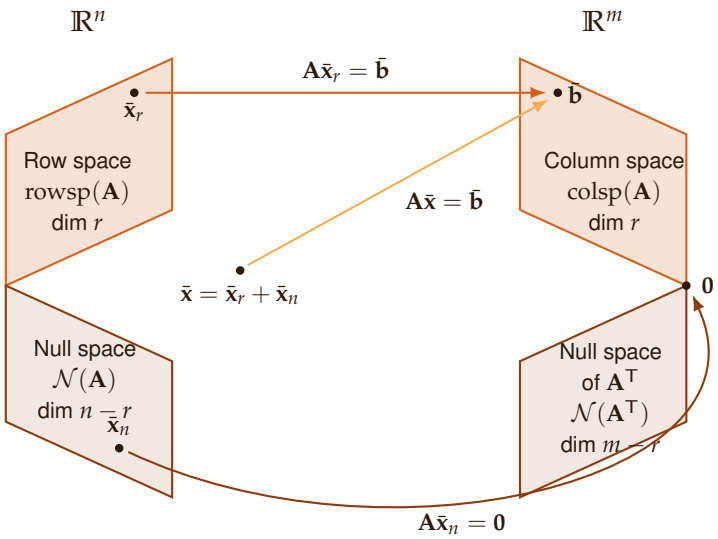


Figure 6.3 The four fundamental subspaces. The row space and null space of $\mathbf{A}$ are orthogonal in $\mathbb{R}^n$; the column space and null space of $\mathbf{A}^\mathsf{T}$ are orthogonal in $\mathbb{R}^m$. $\mathbf{A}$ maps $\bar{\mathbf{x}} = \bar{\mathbf{x}}_r + \bar{\mathbf{x}}_n$ to $\bar{\mathbf{b}}$.

multi-user MIMO, a base station can send to one user without interfering with another by choosing a precoder in the *null space* of the other user's channel. In machine learning, a weight matrix of rank r maps inputs into an r-dimensional column space; low-rank adaptation (LoRA) of large language models trains only such low-rank updates.

Chapter summary

- A vector space is closed under addition and scalar multiplication and obeys the axioms.
- Examples: $\mathbb{K}^n$, polynomials $\mathcal{P}_n(t)$, matrices $\mathbb{K}^{m\times n}$, functions $\mathcal{F}(\mathcal{X})$.
- $\bar{\mathbf{v}}$ is a linear combination if $\bar{\mathbf{v}} = a_1\bar{\mathbf{u}}_1 + \cdots + a_m\bar{\mathbf{u}}_m$; finding the coefficients means solving a linear system.
- A spanning set expresses every vector of $\mathcal{V}$; it is not unique.
- A subspace contains $\mathbf{0}$ and is closed under linear combinations. Examples: lines through O, triangular or symmetric matrices, even polynomials.
- The intersection of subspaces is a subspace; so is the solution set of $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$.
- $\text{span}(\bar{\mathbf{u}}_i)$, the set of all linear combinations, is a subspace.
- Row-equivalent matrices have the same row space; $\text{rowsp}(\mathbf{A}) = \text{colsp}(\mathbf{A}^\mathsf{T})$.
- Linear independence: $a_1\bar{\mathbf{v}}_1 + \cdots + a_n\bar{\mathbf{v}}_n = \mathbf{0}$ only when every $a_i = 0$.
- Vectors are dependent if and only if one is a combination of the others.

- A basis is linearly independent and spans $\mathcal{V}$; all bases have $\dim \mathcal{V}$ elements.
- $\operatorname{rank}(\mathbf{A}) = \dim \operatorname{rowsp}(\mathbf{A}) = \dim \operatorname{colsp}(\mathbf{A})$.
- Nullity is $n - r$, so rank + nullity = number of columns.
- Four fundamental subspaces: the row, column and null spaces of $\mathbf{A}$, and the null space of $\mathbf{A}^\mathsf{T}$.

Exercises

6.1 Show that $\mathcal{W} = \{\begin{bmatrix} a & b & c \end{bmatrix}^\mathsf{T} : a + b + c = 0\}$ is a subspace of $\mathbb{R}^3$, but that $\{\begin{bmatrix} a & b & c \end{bmatrix}^\mathsf{T} : a + b + c = 1\}$ is not.

6.2 Write the vector $\bar{\mathbf{v}} = \begin{bmatrix} 3 & 7 & -4 \end{bmatrix}^\mathsf{T}$ as a linear combination of the vectors $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^\mathsf{T}$, $\begin{bmatrix} 2 & 3 & 7 \end{bmatrix}^\mathsf{T}$ and $\begin{bmatrix} 3 & 5 & 6 \end{bmatrix}^\mathsf{T}$.

6.3 Determine whether $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\mathsf{T}$, $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^\mathsf{T}$ and $\begin{bmatrix} 2 & 5 & 8 \end{bmatrix}^\mathsf{T}$ are linearly independent. If not, write one as a combination of the others.

6.4 Show that the union of two subspaces need not be a subspace.

6.5 For $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix}$, find the rank, and bases of the row space, column space and null space. Verify rank + nullity = 3.

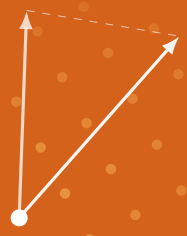
6.6 Show that the $n \times n$ skew-symmetric matrices form a subspace of $\mathbb{R}^{n \times n}$ and find its dimension.

6.7 Find the nullity of the 5×6 matrix $\mathbf{A}$ of Example 6.13.

6.8 Show that every vector in the row space of $\mathbf{A}$ is orthogonal to every vector in the null space of $\mathbf{A}$.

07

Inner Product Spaces



Inner products, norms, orthogonality, Gram–Schmidt, positive definite matrices, normed and complex inner product spaces

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7.1 Inner product spaces

Definition 7.1 (Real inner product)**.** Let $\mathcal{V}$ be a real vector space. Suppose that to each pair of vectors $\bar{\mathbf{u}}, \bar{\mathbf{v}} \in \mathcal{V}$ there is assigned a real number, denoted by $\langle \bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle$. This function is called a (real) **inner product** on $\mathcal{V}$ if it satisfies the following axioms.

1. *Linear property:* $\langle a\bar{\mathbf{u}}_1 + b\bar{\mathbf{u}}_2, \bar{\mathbf{v}} \rangle = a\langle \bar{\mathbf{u}}_1, \bar{\mathbf{v}} \rangle + b\langle \bar{\mathbf{u}}_2, \bar{\mathbf{v}} \rangle$.
2. *Symmetric property:* $\langle \bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle = \langle \bar{\mathbf{v}}, \bar{\mathbf{u}} \rangle$.
3. *Positive definite property:* $\langle \bar{\mathbf{u}}, \bar{\mathbf{u}} \rangle \geq 0$, and $\langle \bar{\mathbf{u}}, \bar{\mathbf{u}} \rangle = 0$ if and only if $\bar{\mathbf{u}} = \mathbf{0}$.

A vector space with an inner product is an **inner product space**.

Using the linearity and symmetry properties, the inner product is also linear in its second argument:

$$\begin{aligned}\langle \bar{\mathbf{u}}, c\bar{\mathbf{v}}_1 + d\bar{\mathbf{v}}_2 \rangle &= \langle c\bar{\mathbf{v}}_1 + d\bar{\mathbf{v}}_2, \bar{\mathbf{u}} \rangle \\ &= c\langle \bar{\mathbf{v}}_1, \bar{\mathbf{u}} \rangle + d\langle \bar{\mathbf{v}}_2, \bar{\mathbf{u}} \rangle \\ &= c\langle \bar{\mathbf{u}}, \bar{\mathbf{v}}_1 \rangle + d\langle \bar{\mathbf{u}}, \bar{\mathbf{v}}_2 \rangle.\end{aligned}$$

Definition 7.2 (Norm induced by an inner product). Since $\langle \bar{\mathbf{u}}, \bar{\mathbf{u}} \rangle \geq 0$, one can define the **norm**, or **length**, of $\bar{\mathbf{u}}$ as

$$\|\bar{\mathbf{u}}\| = \sqrt{\langle \bar{\mathbf{u}}, \bar{\mathbf{u}} \rangle}, \qquad \|\bar{\mathbf{u}}\|^2 = \langle \bar{\mathbf{u}}, \bar{\mathbf{u}} \rangle. \tag{7.1}$$

7.1.1 Examples of inner product spaces

Example 7.1 (Euclidean n-space $\mathbb{R}^n$)

For $\bar{\mathbf{u}} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}^\mathsf{T}$ and $\bar{\mathbf{v}} = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}^\mathsf{T}$, the inner product is the dot product

$$\langle \bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle = \bar{\mathbf{u}} \cdot \bar{\mathbf{v}} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Then $\langle \bar{\mathbf{u}}, \bar{\mathbf{u}} \rangle = a_1^2 + a_2^2 + \cdots + a_n^2$ and

$$\|\bar{\mathbf{u}}\| = \sqrt{\langle \bar{\mathbf{u}}, \bar{\mathbf{u}} \rangle} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}.$$

Definition 7.3 (Function space $C[a,b]$). Denote by $C[a,b]$ the set of all continuous functions on the interval $[a,b]$. An inner product on $C[a,b]$ is

$$\langle f, g \rangle = \int_a^b f(t) g(t)\, dt. \tag{7.2}$$

Example 7.2 (Inner product of functions)

In $C[0,1]$, let $f(t) = 2t + 1$ and $g(t) = t^3$. Find $\langle f, g \rangle$, $\|f\|$ and $\|g\|$.

Solution.

$$\langle f, g \rangle = \int_0^1 (2t+1)t^3\, dt = \left[\tfrac{2}{5}t^5 + \tfrac{1}{4}t^4\right]_0^1 = \tfrac{2}{5} + \tfrac{1}{4} = \tfrac{13}{20}.$$

$$\begin{aligned}\|f\|^2 &= \int_0^1 (2t+1)^2\, dt = \int_0^1 (4t^2 + 4t + 1)\, dt \\ &= \left[\tfrac{4}{3}t^3 + 2t^2 + t\right]_0^1 = \tfrac{13}{3},\end{aligned}$$

so $\|f\| = \sqrt{13/3}$. Similarly,

$$\|g\|^2 = \int_0^1 t^6\, dt = \left[\tfrac{t^7}{7}\right]_0^1 = \tfrac{1}{7}, \qquad \|g\| = \sqrt{\tfrac{1}{7}}.$$

◀

Definition 7.4 (Matrix space). Let $\mathbb{R}^{m\times n}$ denote the vector space of all real $m \times n$ matrices. An inner product on $\mathbb{R}^{m\times n}$ is

$$\langle \mathbf{A}, \mathbf{B}\rangle = \operatorname{tr}(\mathbf{B}^{\mathsf{T}}\mathbf{A}) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ij}. \tag{7.3}$$

The corresponding norm satisfies

$$\langle \mathbf{A}, \mathbf{A}\rangle = \|\mathbf{A}\|^2 = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2;$$

that is, $\|\mathbf{A}\|^2$ is the sum of the squares of the entries of $\mathbf{A}$.

Definition 7.5 (Hilbert space of sequences). Let $\mathcal{H}$ be the set of all infinite sequences $a_1, a_2, a_3, \ldots$ satisfying

$$\sum_{i=1}^{\infty} |a_i|^2 = a_1^2 + a_2^2 + a_3^2 + \cdots < \infty.$$

$\mathcal{H}$ is called the **Hilbert space** of sequences.

Theorem 7.6 ($\mathcal{H}$ is a vector space)

Let $\mathbf{a} = a_1, a_2, a_3, \ldots$ and $\mathbf{b} = b_1, b_2, b_3, \ldots$ belong to $\mathcal{H}$. Then $\alpha\mathbf{a} + \beta\mathbf{b} \in \mathcal{H}$.

Proof. We have $\alpha\mathbf{a} + \beta\mathbf{b} = \alpha a_1 + \beta b_1,\ \alpha a_2 + \beta b_2,\ \ldots$, and

$$\begin{aligned}\sum_{i=1}^{\infty} |\alpha a_i + \beta b_i|^2 &= \sum_{i=1}^{\infty} \left(\alpha^2 a_i^2 + \beta^2 b_i^2 + 2\alpha\beta a_i b_i\right) \\ &\le 2\sum_{i=1}^{\infty} \left(\alpha^2 a_i^2 + \beta^2 b_i^2\right) < \infty,\end{aligned}$$

using $2\alpha\beta a_i b_i \le \alpha^2 a_i^2 + \beta^2 b_i^2$. Hence $\alpha\mathbf{a} + \beta\mathbf{b} \in \mathcal{H}$. ∎

7.2 The Cauchy–Schwarz inequality

Theorem 7.7 (Cauchy–Schwarz inequality)

For all vectors $\bar{\mathbf{u}}, \bar{\mathbf{v}}$ in an inner product space,

$$\langle \bar{\mathbf{u}}, \bar{\mathbf{v}}\rangle^2 \le \langle \bar{\mathbf{u}}, \bar{\mathbf{u}}\rangle\langle \bar{\mathbf{v}}, \bar{\mathbf{v}}\rangle, \qquad \textit{that is,} \qquad |\langle \bar{\mathbf{u}}, \bar{\mathbf{v}}\rangle| \le \|\bar{\mathbf{u}}\|\,\|\bar{\mathbf{v}}\|. \tag{7.4}$$

Proof. If $\bar{\mathbf{v}} = \mathbf{0}$ both sides are zero. Otherwise, for every real t,

$$0 \le \langle \bar{\mathbf{u}} - t\bar{\mathbf{v}}, \bar{\mathbf{u}} - t\bar{\mathbf{v}}\rangle = \|\bar{\mathbf{u}}\|^2 - 2t\langle \bar{\mathbf{u}}, \bar{\mathbf{v}}\rangle + t^2\|\bar{\mathbf{v}}\|^2.$$

A quadratic in t that is never negative has a non-positive discriminant, $4\langle \bar{\mathbf{u}}, \bar{\mathbf{v}}\rangle^2 - 4\|\bar{\mathbf{u}}\|^2\|\bar{\mathbf{v}}\|^2 \le 0$. ∎

Example 7.3 (Cauchy–Schwarz in two spaces)

Solution. In $\mathbb{R}^n$, with $\bar{\mathbf{u}} = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}^\mathsf{T}$ and $\bar{\mathbf{v}} = \begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix}^\mathsf{T}$, the inequality reads

$$(a_1b_1 + \cdots + a_nb_n)^2 \leq (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2).$$

For continuous functions f and g on $[0,1]$, it reads

$$\left|\int_0^1 f(t)g(t)\,dt\right|^2 \leq \int_0^1 f^2(t)\,dt \int_0^1 g^2(t)\,dt, \qquad \text{that is,} \qquad \langle f,g\rangle^2 \leq \|f\|^2\|g\|^2.$$

◀

7.3 Norm properties

Theorem 7.8 (Properties of the norm)

Let $\mathcal{V}$ be an inner product space. The norm in $\mathcal{V}$ satisfies:

1. $\|\bar{\mathbf{v}}\| \geq 0$, *and* $\|\bar{\mathbf{v}}\| = 0 \iff \bar{\mathbf{v}} = \mathbf{0}$;
2. $\|k\bar{\mathbf{v}}\| = |k|\,\|\bar{\mathbf{v}}\|$;
3. $\|\bar{\mathbf{u}} + \bar{\mathbf{v}}\| \leq \|\bar{\mathbf{u}}\| + \|\bar{\mathbf{v}}\|$.

Proof of (3). By Theorem 7.7,

$$\|\bar{\mathbf{u}} + \bar{\mathbf{v}}\|^2 = \|\bar{\mathbf{u}}\|^2 + 2\langle\bar{\mathbf{u}},\bar{\mathbf{v}}\rangle + \|\bar{\mathbf{v}}\|^2 \leq \|\bar{\mathbf{u}}\|^2 + 2\|\bar{\mathbf{u}}\|\|\bar{\mathbf{v}}\| + \|\bar{\mathbf{v}}\|^2 = \left(\|\bar{\mathbf{u}}\| + \|\bar{\mathbf{v}}\|\right)^2.$$

■

7.4 Angle and orthogonality

By the Cauchy–Schwarz inequality,

$$-\|\bar{\mathbf{u}}\|\,\|\bar{\mathbf{v}}\| \leq \langle\bar{\mathbf{u}},\bar{\mathbf{v}}\rangle \leq \|\bar{\mathbf{u}}\|\,\|\bar{\mathbf{v}}\| \;\Rightarrow\; -1 \leq \frac{\langle\bar{\mathbf{u}},\bar{\mathbf{v}}\rangle}{\|\bar{\mathbf{u}}\|\,\|\bar{\mathbf{v}}\|} \leq 1.$$

Definition 7.9 (Angle). The **angle** θ between nonzero vectors $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ in an inner product space is defined by

$$\cos\theta = \frac{\langle\bar{\mathbf{u}},\bar{\mathbf{v}}\rangle}{\|\bar{\mathbf{u}}\|\,\|\bar{\mathbf{v}}\|}. \tag{7.5}$$

Example 7.4 (Angles)

Solution. For the vectors $\begin{bmatrix}3 & -1 & 2\end{bmatrix}^\mathsf{T}$ and $\begin{bmatrix}2 & 4 & 1\end{bmatrix}^\mathsf{T}$ in $\mathbb{R}^3$,

$$\cos\theta = \frac{6-4+2}{\sqrt{14}\sqrt{21}} = \frac{4}{\sqrt{14}\sqrt{21}}.$$

For the functions $f(t) = 2t+1$ and $g(t) = t^3$ of Example 7.2,

$$\cos\theta = \frac{\langle f, g\rangle}{\|f\|\,\|g\|} = \frac{13/20}{\sqrt{13/3}\,\sqrt{1/7}} = \frac{\sqrt{273}}{20}.$$

◂

Definition 7.10 (Orthogonality). Let $\mathcal{V}$ be an inner product space. Vectors $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ are **orthogonal** if

$$\langle \bar{\mathbf{u}}, \bar{\mathbf{v}}\rangle = 0, \qquad \text{equivalently} \qquad \cos\theta = 0.$$

Example 7.5 (Orthogonality)

Solution. For $\bar{\mathbf{u}} = \begin{bmatrix}2 & 1 & -1\end{bmatrix}^\mathsf{T}$ and $\bar{\mathbf{v}} = \begin{bmatrix}1 & -3 & -1\end{bmatrix}^\mathsf{T}$, the dot product is

$$\langle \bar{\mathbf{u}}, \bar{\mathbf{v}}\rangle = 2 - 3 + 1 = 0,$$

so they are orthogonal. For $\sin t$ and $\cos t$ on $[-\pi, \pi]$, the inner product is

$$\int_{-\pi}^{\pi} \sin t \cos t\, dt = \frac{1}{2}\sin^2 t\,\Big|_{-\pi}^{\pi} = 0,$$

so they are orthogonal as well. ◂

7.5 Orthogonal complements

Definition 7.11 (Orthogonal complement). Let $\mathcal{S}$ be a subset of an inner product space $\mathcal{V}$. The **orthogonal complement** of $\mathcal{S}$, denoted $\mathcal{S}^\perp$, consists of the vectors that are orthogonal to every vector in $\mathcal{S}$:

$$\mathcal{S}^\perp = \{\bar{\mathbf{v}} \in \mathcal{V} : \langle \bar{\mathbf{v}}, \bar{\mathbf{u}}\rangle = 0 \text{ for every } \bar{\mathbf{u}} \in \mathcal{S}\}. \tag{7.6}$$

In particular, for $\mathcal{S} = \{\bar{\mathbf{u}}\}$, $\bar{\mathbf{u}}^\perp = \{\bar{\mathbf{v}} \in \mathcal{V} : \langle \bar{\mathbf{v}}, \bar{\mathbf{u}}\rangle = 0\}$.

Theorem 7.12 ($\mathcal{S}^\perp$ is a subspace)

For any subset $\mathcal{S}$ of $\mathcal{V}$, $\mathcal{S}^\perp$ is a vector space.

Proof. $\mathbf{0} \in \mathcal{S}^{\perp}$, since $\langle \mathbf{0}, \bar{\mathbf{u}} \rangle = 0$ for every $\bar{\mathbf{u}} \in \mathcal{S}$. Further, if $\bar{\mathbf{v}}, \bar{\mathbf{w}} \in \mathcal{S}^{\perp}$, then $\langle \bar{\mathbf{v}}, \bar{\mathbf{u}} \rangle = 0$ and $\langle \bar{\mathbf{w}}, \bar{\mathbf{u}} \rangle = 0$ for every $\bar{\mathbf{u}} \in \mathcal{S}$. Then

$$\langle \alpha\bar{\mathbf{v}} + \beta\bar{\mathbf{w}}, \bar{\mathbf{u}} \rangle = \alpha\langle \bar{\mathbf{v}}, \bar{\mathbf{u}} \rangle + \beta\langle \bar{\mathbf{w}}, \bar{\mathbf{u}} \rangle = 0 \quad \text{for every } \bar{\mathbf{u}} \in \mathcal{S}.$$

Therefore $\alpha\bar{\mathbf{v}} + \beta\bar{\mathbf{w}} \in \mathcal{S}^{\perp}$, and $\mathcal{S}^{\perp}$ is a vector space. ■

Geometrically, if $\bar{\mathbf{u}}$ is a nonzero vector in $\mathbb{R}^3$, then $\bar{\mathbf{u}}^{\perp}$ is the plane through the origin O perpendicular to $\bar{\mathbf{u}}$ (Figure 7.1).

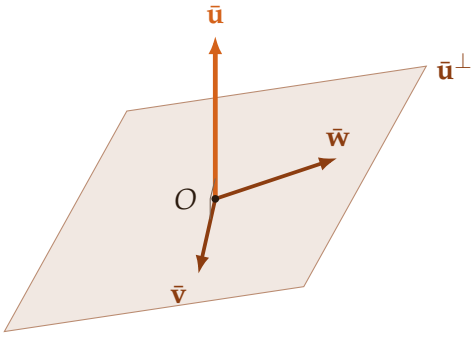


Figure 7.1 The orthogonal complement $\bar{\mathbf{u}}^{\perp}$ of a nonzero $\bar{\mathbf{u}} \in \mathbb{R}^3$ is the plane through the origin perpendicular to $\bar{\mathbf{u}}$; every $\bar{\mathbf{v}}, \bar{\mathbf{w}}$ in it satisfies $\langle \bar{\mathbf{v}}, \bar{\mathbf{u}} \rangle = \langle \bar{\mathbf{w}}, \bar{\mathbf{u}} \rangle = 0$.

Theorem 7.13 (Null space as an orthogonal complement)

Let $\mathcal{W}$ be the solution space of an $m \times n$ homogeneous system $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$. Each solution vector $\bar{\mathbf{x}} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^{\mathsf{T}}$ is orthogonal to each row of $\mathbf{A}$. Hence $\mathcal{W}$, the null space of $\mathbf{A}$, is the orthogonal complement of the row space of $\mathbf{A}$.

Example 7.6 (A basis of $\bar{\mathbf{u}}^{\perp}$)

Find a basis of the subspace $\bar{\mathbf{u}}^{\perp}$ of $\mathbb{R}^3$, where $\bar{\mathbf{u}} = \begin{bmatrix} 1 & -2 & 5 \end{bmatrix}^{\mathsf{T}}$.

Solution. $\bar{\mathbf{u}}^{\perp}$ consists of the vectors $\bar{\mathbf{x}} = \begin{bmatrix} x & y & z \end{bmatrix}^{\mathsf{T}}$ with $\langle \bar{\mathbf{x}}, \bar{\mathbf{u}} \rangle = 0$:

$$\begin{bmatrix} 1 & -2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x - 2y + 5z = 0 \;\Rightarrow\; x = 2y - 5z.$$

Therefore

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2y - 5z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix},$$

and a basis of $\bar{\mathbf{u}}^{\perp}$ is $\begin{bmatrix} 2 & 1 & 0 \end{bmatrix}^{\mathsf{T}}, \begin{bmatrix} -5 & 0 & 1 \end{bmatrix}^{\mathsf{T}}$. ◂

7.6 Orthogonal sets and bases

Definition 7.14 (Orthogonal and orthonormal sets). A set $\mathcal{S} = \{\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \ldots, \bar{\mathbf{u}}_r\}$ is **orthogonal** if each pair of distinct vectors in $\mathcal{S}$ is orthogonal,

$$\langle \bar{\mathbf{u}}_i, \bar{\mathbf{u}}_j \rangle = 0, \quad i \neq j.$$

It is **orthonormal** if it is orthogonal and each vector has unit length:

$$\langle \bar{\mathbf{u}}_i, \bar{\mathbf{u}}_j \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Theorem 7.15 (Orthogonal sets are independent)

An orthogonal set of nonzero vectors is linearly independent.

Theorem 7.16 (Pythagorean theorem)

If $\{\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \ldots, \bar{\mathbf{u}}_r\}$ *is an orthogonal set, then*

$$\|\bar{\mathbf{u}}_1 + \bar{\mathbf{u}}_2 + \cdots + \bar{\mathbf{u}}_r\|^2 = \|\bar{\mathbf{u}}_1\|^2 + \|\bar{\mathbf{u}}_2\|^2 + \cdots + \|\bar{\mathbf{u}}_r\|^2. \tag{7.7}$$

Proof. Expanding the inner product and separating the terms with $i = j$ and $i \neq j$,

$$\begin{aligned} \|\bar{\mathbf{u}}_1 + \bar{\mathbf{u}}_2 + \cdots + \bar{\mathbf{u}}_r\|^2 &= \left\langle \sum_{i=1}^{r} \bar{\mathbf{u}}_i, \sum_{j=1}^{r} \bar{\mathbf{u}}_j \right\rangle \\ &= \sum_{i=1}^{r} \langle \bar{\mathbf{u}}_i, \bar{\mathbf{u}}_i \rangle + \sum_{i=1}^{r} \sum_{\substack{j=1 \\ j \neq i}}^{r} \underbrace{\langle \bar{\mathbf{u}}_i, \bar{\mathbf{u}}_j \rangle}_{0} = \sum_{i=1}^{r} \|\bar{\mathbf{u}}_i\|^2, \end{aligned}$$

since $\langle \bar{\mathbf{u}}_i, \bar{\mathbf{u}}_j \rangle = 0$ for $i \neq j$. ∎

Example 7.7 (The usual basis of $\mathbb{R}^3$)

For $\bar{\mathbf{e}}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\mathsf{T}$, $\bar{\mathbf{e}}_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^\mathsf{T}$, $\bar{\mathbf{e}}_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\mathsf{T}$,

$$\langle \bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2 \rangle = \langle \bar{\mathbf{e}}_1, \bar{\mathbf{e}}_3 \rangle = \langle \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3 \rangle = 0, \qquad \langle \bar{\mathbf{e}}_1, \bar{\mathbf{e}}_1 \rangle = \langle \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_2 \rangle = \langle \bar{\mathbf{e}}_3, \bar{\mathbf{e}}_3 \rangle = 1.$$

Thus $\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3$ form an orthonormal basis of $\mathbb{R}^3$.

Example 7.8 (An orthogonal set of functions)

Let $\mathcal{V} = \mathcal{C}[-\pi, \pi]$, the continuous functions on $-\pi \leq t \leq \pi$, with inner product $\langle f, g \rangle = \int_{-\pi}^{\pi} f(t) g(t)\, dt$. A classical orthogonal set in $\mathcal{V}$ is

$$\{1, \cos t, \cos 2t, \cos 3t, \ldots, \sin t, \sin 2t, \sin 3t, \ldots\}.$$

Solution. For example, consider $\langle \sin kt, \cos lt \rangle = \int_{-\pi}^{\pi} \sin kt \cos lt\, dt$. For $k \neq l$, using

$$2\sin kt\cos lt = \sin(k+l)t + \sin(k-l)t,$$

$$\int_{-\pi}^{\pi} \sin kt\cos lt\,dt = -\frac{1}{2}\left[\frac{\cos(k+l)t}{k+l} + \frac{\cos(k-l)t}{k-l}\right]_{-\pi}^{\pi} = 0,$$

since cosine takes the same value at $\pm\pi$. ◂

Theorem 7.17 (Coordinates in an orthogonal basis)

Let $\{\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \ldots, \bar{\mathbf{u}}_n\}$ be an orthogonal basis of $\mathcal{V}$. Then any $\bar{\mathbf{v}} \in \mathcal{V}$ can be written as

$$\bar{\mathbf{v}} = a_1\bar{\mathbf{u}}_1 + a_2\bar{\mathbf{u}}_2 + \cdots + a_n\bar{\mathbf{u}}_n, \qquad \textit{where} \qquad a_i = \frac{\langle \bar{\mathbf{v}}, \bar{\mathbf{u}}_i\rangle}{\langle \bar{\mathbf{u}}_i, \bar{\mathbf{u}}_i\rangle}.$$

Proof. Taking the inner product of $\bar{\mathbf{v}}$ with $\bar{\mathbf{u}}_i$,

$$\begin{aligned}\langle \bar{\mathbf{v}}, \bar{\mathbf{u}}_i\rangle &= a_1\langle \bar{\mathbf{u}}_1, \bar{\mathbf{u}}_i\rangle + a_2\langle \bar{\mathbf{u}}_2, \bar{\mathbf{u}}_i\rangle + \cdots + a_n\langle \bar{\mathbf{u}}_n, \bar{\mathbf{u}}_i\rangle \\ &= a_i\langle \bar{\mathbf{u}}_i, \bar{\mathbf{u}}_i\rangle \;\Rightarrow\; a_i = \frac{\langle \bar{\mathbf{v}}, \bar{\mathbf{u}}_i\rangle}{\langle \bar{\mathbf{u}}_i, \bar{\mathbf{u}}_i\rangle}.\end{aligned}$$

■

7.7 Projection

Definition 7.18 (Projection along a vector). Let $\mathcal{V}$ be an inner product space. The **projection** of $\bar{\mathbf{v}}$ along $\bar{\mathbf{w}}$ is

$$\frac{\langle \bar{\mathbf{v}}, \bar{\mathbf{w}}\rangle}{\langle \bar{\mathbf{w}}, \bar{\mathbf{w}}\rangle}\bar{\mathbf{w}}, \tag{7.8}$$

and the component of $\bar{\mathbf{v}}$ orthogonal to $\bar{\mathbf{w}}$ is

$$\bar{\mathbf{v}} - \frac{\langle \bar{\mathbf{v}}, \bar{\mathbf{w}}\rangle}{\langle \bar{\mathbf{w}}, \bar{\mathbf{w}}\rangle}\bar{\mathbf{w}}.$$

The remainder is orthogonal to $\bar{\mathbf{w}}$:

$$\begin{aligned}\left\langle \bar{\mathbf{v}} - \frac{\langle \bar{\mathbf{v}}, \bar{\mathbf{w}}\rangle}{\langle \bar{\mathbf{w}}, \bar{\mathbf{w}}\rangle}\bar{\mathbf{w}},\ \bar{\mathbf{w}}\right\rangle &= \langle \bar{\mathbf{v}}, \bar{\mathbf{w}}\rangle - \frac{\langle \bar{\mathbf{v}}, \bar{\mathbf{w}}\rangle}{\langle \bar{\mathbf{w}}, \bar{\mathbf{w}}\rangle}\langle \bar{\mathbf{w}}, \bar{\mathbf{w}}\rangle \\ &= \langle \bar{\mathbf{v}}, \bar{\mathbf{w}}\rangle - \langle \bar{\mathbf{v}}, \bar{\mathbf{w}}\rangle = 0.\end{aligned}$$

Definition 7.19 (Projection onto a subspace). Suppose $\mathcal{W} = \{\bar{\mathbf{w}}_1, \bar{\mathbf{w}}_2, \ldots, \bar{\mathbf{w}}_r\}$ is an orthogonal set. The projection of $\bar{\mathbf{v}}$ onto the space spanned by $\mathcal{W}$ is

$$\frac{\langle \bar{\mathbf{v}}, \bar{\mathbf{w}}_1\rangle}{\langle \bar{\mathbf{w}}_1, \bar{\mathbf{w}}_1\rangle}\bar{\mathbf{w}}_1 + \cdots + \frac{\langle \bar{\mathbf{v}}, \bar{\mathbf{w}}_r\rangle}{\langle \bar{\mathbf{w}}_r, \bar{\mathbf{w}}_r\rangle}\bar{\mathbf{w}}_r,$$

and the component of $\bar{\mathbf{v}}$ orthogonal to it is

$$\bar{\mathbf{v}} - \frac{\langle \bar{\mathbf{v}}, \bar{\mathbf{w}}_1 \rangle}{\langle \bar{\mathbf{w}}_1, \bar{\mathbf{w}}_1 \rangle} \bar{\mathbf{w}}_1 - \cdots - \frac{\langle \bar{\mathbf{v}}, \bar{\mathbf{w}}_r \rangle}{\langle \bar{\mathbf{w}}_r, \bar{\mathbf{w}}_r \rangle} \bar{\mathbf{w}}_r.$$

7.8 Gram–Schmidt orthogonalization

Suppose $\{\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \ldots, \bar{\mathbf{v}}_n\}$ is a basis of an inner product space $\mathcal{V}$. One can use it to construct an orthogonal basis $\{\bar{\mathbf{w}}_1, \bar{\mathbf{w}}_2, \ldots, \bar{\mathbf{w}}_n\}$ as follows:

$$\begin{aligned}
\bar{\mathbf{w}}_1 &= \bar{\mathbf{v}}_1, \\
\bar{\mathbf{w}}_2 &= \bar{\mathbf{v}}_2 - \frac{\langle \bar{\mathbf{v}}_2, \bar{\mathbf{w}}_1 \rangle}{\langle \bar{\mathbf{w}}_1, \bar{\mathbf{w}}_1 \rangle} \bar{\mathbf{w}}_1, \\
\bar{\mathbf{w}}_3 &= \bar{\mathbf{v}}_3 - \frac{\langle \bar{\mathbf{v}}_3, \bar{\mathbf{w}}_1 \rangle}{\langle \bar{\mathbf{w}}_1, \bar{\mathbf{w}}_1 \rangle} \bar{\mathbf{w}}_1 - \frac{\langle \bar{\mathbf{v}}_3, \bar{\mathbf{w}}_2 \rangle}{\langle \bar{\mathbf{w}}_2, \bar{\mathbf{w}}_2 \rangle} \bar{\mathbf{w}}_2, \\
&\vdots \\
\bar{\mathbf{w}}_n &= \bar{\mathbf{v}}_n - \frac{\langle \bar{\mathbf{v}}_n, \bar{\mathbf{w}}_1 \rangle}{\langle \bar{\mathbf{w}}_1, \bar{\mathbf{w}}_1 \rangle} \bar{\mathbf{w}}_1 - \frac{\langle \bar{\mathbf{v}}_n, \bar{\mathbf{w}}_2 \rangle}{\langle \bar{\mathbf{w}}_2, \bar{\mathbf{w}}_2 \rangle} \bar{\mathbf{w}}_2 - \cdots - \frac{\langle \bar{\mathbf{v}}_n, \bar{\mathbf{w}}_{n-1} \rangle}{\langle \bar{\mathbf{w}}_{n-1}, \bar{\mathbf{w}}_{n-1} \rangle} \bar{\mathbf{w}}_{n-1}.
\end{aligned}$$

This is the **Gram–Schmidt orthogonalization process**. The component of $\bar{\mathbf{v}}_k$ along $\bar{\mathbf{w}}_i$ is

$$\frac{\langle \bar{\mathbf{v}}_k, \bar{\mathbf{w}}_i \rangle}{\langle \bar{\mathbf{w}}_i, \bar{\mathbf{w}}_i \rangle} \bar{\mathbf{w}}_i.$$

Removing all the components along $\bar{\mathbf{w}}_1, \bar{\mathbf{w}}_2, \ldots, \bar{\mathbf{w}}_{k-1}$ leaves the remainder $\bar{\mathbf{w}}_k$, which is therefore orthogonal to $\bar{\mathbf{w}}_1, \bar{\mathbf{w}}_2, \ldots, \bar{\mathbf{w}}_{k-1}$ (Figure 7.2).

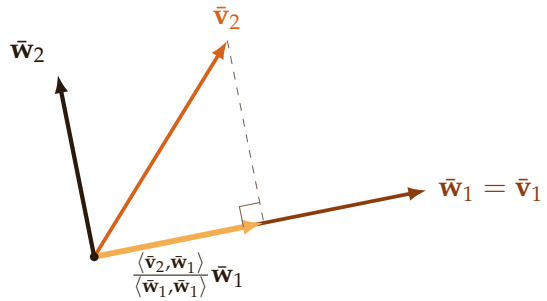


Figure 7.2 One Gram–Schmidt step: $\bar{\mathbf{w}}_2$ is $\bar{\mathbf{v}}_2$ minus its projection along $\bar{\mathbf{w}}_1$, and is orthogonal to $\bar{\mathbf{w}}_1$.

Example 7.9 (Gram–Schmidt in $\mathbb{R}^4$)

Find an orthogonal basis, and then an orthonormal basis, of the subspace $\mathcal{U}$ of $\mathbb{R}^4$ spanned by

$$\bar{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \bar{\mathbf{v}}_2 = \begin{bmatrix} 1 \\ 4 \\ 2 \\ 5 \end{bmatrix}, \qquad \bar{\mathbf{v}}_3 = \begin{bmatrix} 1 \\ -3 \\ 0 \\ -2 \end{bmatrix}.$$

Solution. Set $\bar{\mathbf{w}}_1 = \bar{\mathbf{v}}_1 = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^{\mathsf{T}}$. Next,

$$\bar{\mathbf{w}}_2 = \bar{\mathbf{v}}_2 - \frac{\langle \bar{\mathbf{v}}_2, \bar{\mathbf{w}}_1\rangle}{\langle \bar{\mathbf{w}}_1, \bar{\mathbf{w}}_1\rangle}\bar{\mathbf{w}}_1 = \begin{bmatrix}1\\4\\2\\5\end{bmatrix} - \frac{12}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}-2\\1\\-1\\2\end{bmatrix}.$$

Now

$$\begin{aligned}\bar{\mathbf{w}}_3 &= \bar{\mathbf{v}}_3 - \frac{\langle \bar{\mathbf{v}}_3, \bar{\mathbf{w}}_1\rangle}{\langle \bar{\mathbf{w}}_1, \bar{\mathbf{w}}_1\rangle}\bar{\mathbf{w}}_1 - \frac{\langle \bar{\mathbf{v}}_3, \bar{\mathbf{w}}_2\rangle}{\langle \bar{\mathbf{w}}_2, \bar{\mathbf{w}}_2\rangle}\bar{\mathbf{w}}_2 \\ &= \begin{bmatrix}1\\-3\\0\\-2\end{bmatrix} - \frac{-4}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{-9}{10}\begin{bmatrix}-2\\1\\-1\\2\end{bmatrix} = \begin{bmatrix}2\\-2\\1\\-1\end{bmatrix} + \frac{9}{10}\begin{bmatrix}-2\\1\\-1\\2\end{bmatrix} = \frac{1}{10}\begin{bmatrix}2\\-11\\1\\8\end{bmatrix}.\end{aligned}$$

Normalizing $\bar{\mathbf{w}}_1$, $\bar{\mathbf{w}}_2$ and $\bar{\mathbf{w}}_3$ gives an orthonormal basis:

$$\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \qquad \frac{1}{\sqrt{10}}\begin{bmatrix}-2\\1\\-1\\2\end{bmatrix}, \qquad \frac{1}{\sqrt{190}}\begin{bmatrix}2\\-11\\1\\8\end{bmatrix}.$$

◀

Example 7.10 (Gram–Schmidt on polynomials)

Let $\mathcal{V}$ be the vector space of polynomials with inner product $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\,dt$. Apply the Gram–Schmidt process to $\{1, t, t^2, t^3\}$ to find an orthogonal basis $\{f_0, f_1, f_2, f_3\}$.

Solution. Set $f_0 = 1$. Then

$$f_1 = t - \frac{\langle t, 1\rangle}{\langle 1, 1\rangle}1 = t - \frac{1}{2}\int_{-1}^{1} t\,dt = t - 0 = t.$$

Next,

$$\begin{aligned}f_2 &= t^2 - \frac{\langle t^2, 1\rangle}{\langle 1, 1\rangle}1 - \frac{\langle t^2, t\rangle}{\langle t, t\rangle}t = t^2 - \frac{1}{2}\int_{-1}^{1} t^2\,dt - \frac{t}{2/3}\int_{-1}^{1} t^3\,dt \\ &= t^2 - \frac{2/3}{2} = t^2 - \frac{1}{3} \;\sim\; 3t^2 - 1.\end{aligned}$$

Finally,

$$\begin{aligned}f_3 &= t^3 - \frac{\langle t^3, 1\rangle}{\langle 1, 1\rangle}1 - \frac{\langle t^3, t\rangle}{\langle t, t\rangle}t - \frac{\langle t^3, 3t^2 - 1\rangle}{\langle 3t^2 - 1, 3t^2 - 1\rangle}(3t^2 - 1) \\ &= t^3 - 0 - \frac{t}{2/3}\int_{-1}^{1} t^4\,dt - 0 = t^3 - \frac{2/5}{2/3}t = t^3 - \frac{3}{5}t \;\sim\; 5t^3 - 3t.\end{aligned}$$

Here $\sim$ means equal up to a nonzero scale factor, which does not affect orthogonality. Rescaling so that $p(1) = 1$ gives

$$1, \qquad t, \qquad \tfrac{1}{2}(3t^2 - 1), \qquad \tfrac{1}{2}(5t^3 - 3t),$$

the first four **Legendre polynomials**. ◂

7.9 Orthogonal matrices

An **orthogonal matrix** $\mathbf{P}$ satisfies $\mathbf{P}\mathbf{P}^\mathsf{T} = \mathbf{P}^\mathsf{T}\mathbf{P} = \mathbf{I}$.

Theorem 7.20 (Characterizations of orthogonal matrices)

Let $\mathbf{P}$ be a real matrix. The following statements are equivalent:

1. *$\mathbf{P}$ is orthogonal;*
2. *the rows of $\mathbf{P}$ form an orthonormal set;*
3. *the columns of $\mathbf{P}$ form an orthonormal set.*

Theorem 7.21 (Orthonormal bases and orthogonal matrices)

Consider the matrices whose columns are two orthonormal bases,

$$\mathbf{E} = \begin{bmatrix} | & | & & | \\ \bar{\mathbf{e}}_1 & \bar{\mathbf{e}}_2 & \cdots & \bar{\mathbf{e}}_n \\ | & | & & | \end{bmatrix}, \qquad \tilde{\mathbf{E}} = \begin{bmatrix} | & | & & | \\ \tilde{\mathbf{e}}_1 & \tilde{\mathbf{e}}_2 & \cdots & \tilde{\mathbf{e}}_n \\ | & | & & | \end{bmatrix}.$$

1. *If $\mathbf{P}$ is the change-of-basis matrix from $\mathbf{E}$ to $\tilde{\mathbf{E}}$, that is, $\tilde{\mathbf{E}} = \mathbf{E}\mathbf{P}$, then $\mathbf{P}$ is orthogonal.*
2. *Conversely, suppose the columns of $\mathbf{E}$ are an orthonormal basis of an n-dimensional space $\mathcal{V}$, and $\mathbf{P}$ is an orthogonal matrix. Then the columns of $\tilde{\mathbf{E}} = \mathbf{E}\mathbf{P}$ also form an orthonormal basis of $\mathcal{V}$.*

7.10 Positive definite and semidefinite matrices

Definition 7.22 (Positive definite and semidefinite). Let $\mathbf{A}$ be a symmetric matrix, $\mathbf{A} = \mathbf{A}^\mathsf{T}$. Then $\mathbf{A}$ is **positive definite** (PD) if, for every nonzero vector $\bar{\mathbf{u}} \in \mathbb{R}^n$,

$$\langle \bar{\mathbf{u}}, \mathbf{A}\bar{\mathbf{u}} \rangle = \bar{\mathbf{u}}^\mathsf{T}\mathbf{A}\bar{\mathbf{u}} > 0, \tag{7.9}$$

and **positive semidefinite** (PSD) if, for every nonzero vector $\bar{\mathbf{u}} \in \mathbb{R}^n$,

$$\langle \bar{\mathbf{u}}, \mathbf{A}\bar{\mathbf{u}} \rangle = \bar{\mathbf{u}}^\mathsf{T}\mathbf{A}\bar{\mathbf{u}} \geq 0.$$

Theorem 7.23 (The 2×2 case)

The symmetric matrix $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$ *is*

- *positive definite if and only if* $a > 0$, $d > 0$ *and* $ad - b^2 > 0$;
- *positive semidefinite if and only if* $a \geq 0$, $d \geq 0$ *and* $ad - b^2 \geq 0$.

Example 7.11 (A positive definite matrix)

The matrix $\begin{bmatrix} 2 & -3 \\ -3 & 7 \end{bmatrix}$ is positive definite, since $a = 2 > 0$, $d = 7 > 0$ and

$$ad - b^2 = 14 - 9 = 5 > 0.$$

Theorem 7.24 (Inner product from a PD matrix)

Let $\mathbf{A}$ *be a real positive definite matrix. Then*

$$\langle \bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle = \bar{\mathbf{u}}^{\mathsf{T}} \mathbf{A} \bar{\mathbf{v}}$$

is an inner product on $\mathbb{R}^n$, *with* $\langle \bar{\mathbf{u}}, \bar{\mathbf{u}} \rangle = \|\bar{\mathbf{u}}\|^2 = \bar{\mathbf{u}}^{\mathsf{T}} \mathbf{A} \bar{\mathbf{u}}$.

7.11 Normed vector spaces

Definition 7.25 (Norm). Let $\mathcal{V}$ be a vector space, and suppose that to each $\bar{\mathbf{v}} \in \mathcal{V}$ there is assigned a real number, denoted $\|\bar{\mathbf{v}}\|$. This is called a **norm** on $\mathcal{V}$ if it satisfies:

1. $\|\bar{\mathbf{v}}\| \geq 0$, and $\|\bar{\mathbf{v}}\| = 0$ if and only if $\bar{\mathbf{v}} = \mathbf{0}$;
2. $\|k\bar{\mathbf{v}}\| = |k| \, \|\bar{\mathbf{v}}\|$;
3. $\|\bar{\mathbf{u}} + \bar{\mathbf{v}}\| \leq \|\bar{\mathbf{u}}\| + \|\bar{\mathbf{v}}\|$.

A vector space with a norm is a **normed vector space**.

Definition 7.26 (Distance). Let $\mathcal{V}$ be a normed vector space. The **distance function** is

$$d(\bar{\mathbf{u}}, \bar{\mathbf{v}}) = \|\bar{\mathbf{u}} - \bar{\mathbf{v}}\|. \tag{7.10}$$

It satisfies the three axioms:

1. $d(\bar{\mathbf{u}}, \bar{\mathbf{v}}) \geq 0$, and $d(\bar{\mathbf{u}}, \bar{\mathbf{v}}) = 0$ if and only if $\bar{\mathbf{u}} = \bar{\mathbf{v}}$;
2. $d(\bar{\mathbf{u}}, \bar{\mathbf{v}}) = d(\bar{\mathbf{v}}, \bar{\mathbf{u}})$;
3. $d(\bar{\mathbf{u}}, \bar{\mathbf{v}}) \leq d(\bar{\mathbf{u}}, \bar{\mathbf{w}}) + d(\bar{\mathbf{w}}, \bar{\mathbf{v}})$.

Every inner product space is a normed vector space with $\|\bar{\mathbf{v}}\| = \sqrt{\langle \bar{\mathbf{v}}, \bar{\mathbf{v}} \rangle}$. However, a normed vector space need not be an inner product space.

7.11.1 Three important norms

For $\bar{\mathbf{u}} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}^\mathsf{T}$, three important norms on $\mathbb{R}^n$ and $\mathbb{C}^n$ are:

$$\begin{aligned} \ell_2 \text{ norm:} \quad & \|\bar{\mathbf{u}}\|_2 = \sqrt{|u_1|^2 + |u_2|^2 + \cdots + |u_n|^2}, \\ \ell_1 \text{ norm:} \quad & \|\bar{\mathbf{u}}\|_1 = |u_1| + |u_2| + \cdots + |u_n|, \\ \ell_\infty \text{ norm:} \quad & \|\bar{\mathbf{u}}\|_\infty = \max_i |u_i|. \end{aligned}$$

Example 7.12 (Three norms of one vector)

For $\bar{\mathbf{u}} = \begin{bmatrix} 2 & -6 & 3 \end{bmatrix}^\mathsf{T}$,

$$\begin{aligned} \|\bar{\mathbf{u}}\|_2 &= \sqrt{4 + 36 + 9} = \sqrt{49} = 7, \\ \|\bar{\mathbf{u}}\|_1 &= 2 + 6 + 3 = 11, \\ \|\bar{\mathbf{u}}\|_\infty &= \max\{|2|, |-6|, |3|\} = 6. \end{aligned}$$

7.11.2 Unit balls

Consider the set of points with $\|\bar{\mathbf{u}}\| = 1$ when $n = 2$ (Figure 7.3).

- ℓ_2: $\|\bar{\mathbf{u}}\|_2 = 1 \Rightarrow \sqrt{u_1^2 + u_2^2} = 1$, the circle D_1.
- ℓ_1: $\|\bar{\mathbf{u}}\|_1 = 1 \Rightarrow |u_1| + |u_2| = 1$. In the four quadrants this gives the lines

$$\begin{aligned} Q_1 &: u_1 + u_2 = 1, & Q_2 &: -u_1 + u_2 = 1, \\ Q_3 &: -u_1 - u_2 = 1, & Q_4 &: u_1 - u_2 = 1, \end{aligned}$$

  which form the diamond D_2.
- ℓ_∞: $\|\bar{\mathbf{u}}\|_\infty = 1 \Rightarrow \max\{|u_1|, |u_2|\} = 1$. Either $u_1 = \pm 1$ and $|u_2| \leq 1$, that is $-1 \leq u_2 \leq 1$; or $u_2 = \pm 1$ and $|u_1| \leq 1$, that is $-1 \leq u_1 \leq 1$. This is the square D_3.

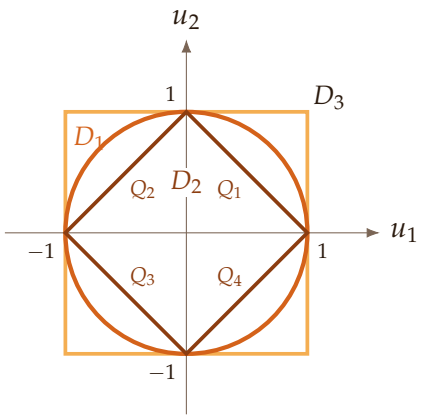


Figure 7.3 Unit balls in $\mathbb{R}^2$: the ℓ_2 circle D_1, the ℓ_1 diamond D_2 (sides Q_1–Q_4) and the ℓ_∞ square D_3.

7.12 Complex inner product spaces

Definition 7.27 (Complex inner product). Consider a vector space $\mathcal{V}$ over the complex field $\mathbb{C}$. Each pair of vectors $\bar{\mathbf{u}}, \bar{\mathbf{v}}$ is assigned a complex number $\langle \bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle$ satisfying:

1. *Linear property:* $\langle \bar{\mathbf{v}}, a\bar{\mathbf{u}}_1 + b\bar{\mathbf{u}}_2 \rangle = a^* \langle \bar{\mathbf{v}}, \bar{\mathbf{u}}_1 \rangle + b^* \langle \bar{\mathbf{v}}, \bar{\mathbf{u}}_2 \rangle$;
2. *Conjugate symmetry property:* $\langle \bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle = \langle \bar{\mathbf{v}}, \bar{\mathbf{u}} \rangle^*$;
3. *Positive definite property:* $\langle \bar{\mathbf{u}}, \bar{\mathbf{u}} \rangle \geq 0$, and $\langle \bar{\mathbf{u}}, \bar{\mathbf{u}} \rangle = 0$ if and only if $\bar{\mathbf{u}} = \mathbf{0}$.

Remark. Combining (1) and (2), the complex inner product is linear in its first argument, $\langle a\bar{\mathbf{u}}_1 + b\bar{\mathbf{u}}_2, \bar{\mathbf{v}} \rangle = a\langle \bar{\mathbf{u}}_1, \bar{\mathbf{v}} \rangle + b\langle \bar{\mathbf{u}}_2, \bar{\mathbf{v}} \rangle$, and conjugate-linear in its second. Conjugate symmetry also makes $\langle \bar{\mathbf{u}}, \bar{\mathbf{u}} \rangle = \langle \bar{\mathbf{u}}, \bar{\mathbf{u}} \rangle^*$ real, so (3) makes sense.

Example 7.13 (Complex inner product spaces)

Solution. *Complex vectors.* For $\bar{\mathbf{u}} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}^{\mathsf{T}}$ and $\bar{\mathbf{v}} = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}^{\mathsf{T}}$ in $\mathbb{C}^n$,

$$\langle \bar{\mathbf{v}}, \bar{\mathbf{u}} \rangle = \bar{\mathbf{u}}^{\mathsf{H}} \bar{\mathbf{v}} = a_1^* b_1 + a_2^* b_2 + \cdots + a_n^* b_n.$$

Complex functions. On the space of complex continuous functions on $a \leq t \leq b$,

$$\langle f, g \rangle = \int_a^b g^*(t) f(t)\, dt.$$

Complex matrices. On the space of $m \times n$ matrices over $\mathbb{C}$,

$$\langle \mathbf{B}, \mathbf{A} \rangle = \operatorname{tr}(\mathbf{A}^{\mathsf{H}} \mathbf{B}) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^* b_{ij}.$$

◂

Theorem 7.28 (Complex Cauchy–Schwarz inequality)

In a complex inner product space, $|\langle \bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle| \leq \|\bar{\mathbf{u}}\| \, \|\bar{\mathbf{v}}\|$.

AIML & 6G Connection

Gram–Schmidt applied to the columns of a matrix gives its QR factorization, $\mathbf{A} = \mathbf{QR}$, used in least squares, in MIMO detection (successive interference cancellation) and in channel estimation. Positive definite matrices appear as covariance matrices $\mathbb{E}[\bar{\mathbf{x}}\bar{\mathbf{x}}^{\mathsf{H}}]$ of signals and as kernel (Gram) matrices in machine learning. The ℓ_1 norm promotes sparse solutions: it is the basis of the LASSO and of compressed sensing for sparse mmWave channels. The complex inner product $\bar{\mathbf{w}}^{\mathsf{H}} \bar{\mathbf{h}}$ is the matched filter of a receiver.

Chapter summary

- An inner product is linear, symmetric and positive definite; the norm is $\|\bar{\mathbf{u}}\| = \sqrt{\langle \bar{\mathbf{u}}, \bar{\mathbf{u}} \rangle}$.
- Examples: the dot product on $\mathbb{R}^n$, $\int fg\,dt$, $\operatorname{tr}(\mathbf{B}^\mathsf{T}\mathbf{A})$, and square-summable sequences.
- Cauchy–Schwarz: $|\langle \bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle| \leq \|\bar{\mathbf{u}}\|\,\|\bar{\mathbf{v}}\|$.
- The norm is non-negative, homogeneous and obeys the triangle inequality.
- Angle: $\cos\theta = \langle \bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle / (\|\bar{\mathbf{u}}\|\|\bar{\mathbf{v}}\|)$; orthogonal when $\langle \bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle = 0$.
- $\mathcal{S}^\perp$ contains every vector orthogonal to all of $\mathcal{S}$; it is a vector space.
- The solutions of $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$ form the orthogonal complement of the row space of $\mathbf{A}$.
- Nonzero orthogonal vectors are linearly independent, and $\|\sum_i \bar{\mathbf{u}}_i\|^2 = \sum_i \|\bar{\mathbf{u}}_i\|^2$.
- In an orthogonal basis, $\bar{\mathbf{v}} = \sum_i a_i \bar{\mathbf{u}}_i$ with $a_i = \langle \bar{\mathbf{v}}, \bar{\mathbf{u}}_i \rangle / \langle \bar{\mathbf{u}}_i, \bar{\mathbf{u}}_i \rangle$.
- The projection of $\bar{\mathbf{v}}$ along $\bar{\mathbf{w}}$ is $\frac{\langle \bar{\mathbf{v}}, \bar{\mathbf{w}} \rangle}{\langle \bar{\mathbf{w}}, \bar{\mathbf{w}} \rangle} \bar{\mathbf{w}}$; the remainder is orthogonal to $\bar{\mathbf{w}}$.
- Gram–Schmidt subtracts these projections to turn any basis into an orthogonal basis.
- An orthogonal matrix satisfies $\mathbf{P}\mathbf{P}^\mathsf{T} = \mathbf{P}^\mathsf{T}\mathbf{P} = \mathbf{I}$; its rows and columns are orthonormal. If the columns of $\mathbf{E}$ are orthonormal and $\mathbf{P}$ is orthogonal, so are the columns of $\mathbf{EP}$.
- A symmetric $\mathbf{A}$ is PD if $\bar{\mathbf{u}}^\mathsf{T}\mathbf{A}\bar{\mathbf{u}} > 0$ for every $\bar{\mathbf{u}} \neq \mathbf{0}$, and PSD if $\bar{\mathbf{u}}^\mathsf{T}\mathbf{A}\bar{\mathbf{u}} \geq 0$.
- A norm is non-negative, homogeneous and obeys the triangle inequality; $d(\bar{\mathbf{u}}, \bar{\mathbf{v}}) = \|\bar{\mathbf{u}} - \bar{\mathbf{v}}\|$.
- The unit balls of the ℓ_2, ℓ_1 and ℓ_∞ norms in $\mathbb{R}^2$ are a circle, a diamond and a square.
- Complex inner product: $\langle \bar{\mathbf{v}}, \bar{\mathbf{u}} \rangle = \bar{\mathbf{u}}^\mathsf{H}\bar{\mathbf{v}}$, with $\langle \bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle = \langle \bar{\mathbf{v}}, \bar{\mathbf{u}} \rangle^*$.

Exercises

7.1 Show that $\langle \bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle = \bar{\mathbf{u}}^\mathsf{T} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \bar{\mathbf{v}}$ is an inner product on $\mathbb{R}^2$.

7.2 In $\mathcal{C}[0,1]$, find $\langle f, g \rangle$, $\|f\|$, $\|g\|$ and $\cos\theta$ for $f(t) = t$ and $g(t) = t^2$.

7.3 Let $\bar{\mathbf{a}} = \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}^\mathsf{T}$ and $\bar{\mathbf{b}} = \begin{bmatrix} 0 & 1 & 1 & -1 \end{bmatrix}^\mathsf{T}$. Find a basis of the orthogonal complement of $\operatorname{span}(\bar{\mathbf{a}}, \bar{\mathbf{b}})$ in $\mathbb{R}^4$.

7.4 Apply Gram–Schmidt to $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\mathsf{T}$, $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^\mathsf{T}$, $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\mathsf{T}$.

7.5 Compute the ℓ_1, ℓ_2 and ℓ_∞ norms of $\begin{bmatrix} 3 & -4 & 12 \end{bmatrix}^\mathsf{T}$.

7.6 Show that $\|\bar{\mathbf{u}}\|_\infty \leq \|\bar{\mathbf{u}}\|_2 \leq \|\bar{\mathbf{u}}\|_1$ for every $\bar{\mathbf{u}} \in \mathbb{R}^n$.

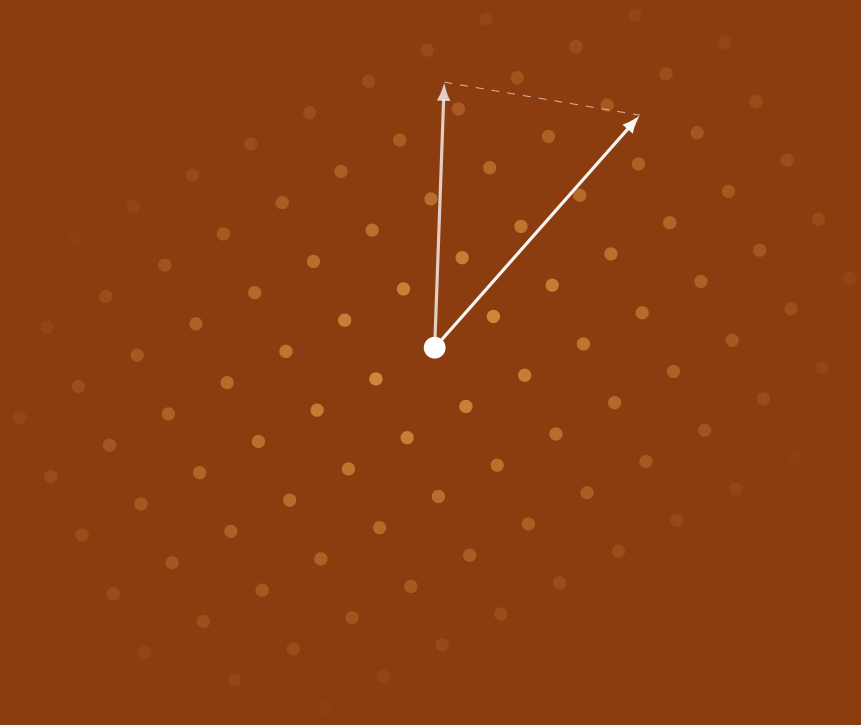
7.7 Show that a norm induced by an inner product satisfies the parallelogram law $\|\bar{\mathbf{u}} + \bar{\mathbf{v}}\|^2 + \|\bar{\mathbf{u}} - \bar{\mathbf{v}}\|^2 = 2\|\bar{\mathbf{u}}\|^2 + 2\|\bar{\mathbf{v}}\|^2$, and use $\bar{\mathbf{u}} = \bar{\mathbf{e}}_1$, $\bar{\mathbf{v}} = \bar{\mathbf{e}}_2$ to show that the ℓ_1 norm does not come from an inner product.

7.8 Is $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ positive definite? Justify your answer.

7.9 Show that $\sin t$ and $\sin 2t$ are orthogonal in $\mathcal{C}[-\pi, \pi]$.

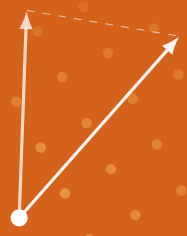
PART V

Eigenvalue And Singular Value Decompositions



08

Eigenvalue Decomposition



Eigenvalues, eigenvectors, the eigenvalue decomposition and its properties

In this chapter

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8.1 Eigenvalues and eigenvectors

Definition 8.1 (Eigenvalue and eigenvector)**.** Let $\mathbf{A}$ be an $n \times n$ matrix. A scalar λ is called an **eigenvalue** of $\mathbf{A}$ if there is a nonzero vector $\bar{\mathbf{v}} \neq \mathbf{0}$, called an **eigenvector**, such that

$$\mathbf{A}\bar{\mathbf{v}} = \lambda\bar{\mathbf{v}}. \tag{8.1}$$

An eigenvector is a direction that $\mathbf{A}$ does not turn: $\mathbf{A}$ only stretches it by the factor λ (Figure 8.1).

We can rewrite (8.1) as

$$\mathbf{A}\bar{\mathbf{v}} = \lambda\mathbf{I}\bar{\mathbf{v}} \;\Rightarrow\; (\mathbf{A} - \lambda\mathbf{I})\bar{\mathbf{v}} = \mathbf{0}.$$

This homogeneous system has a nonzero solution $\bar{\mathbf{v}}$ exactly when $\mathbf{A} - \lambda\mathbf{I}$ is singular. Therefore λ is an eigenvalue of $\mathbf{A}$ if and only if $\mathbf{A} - \lambda\mathbf{I}$ is singular.

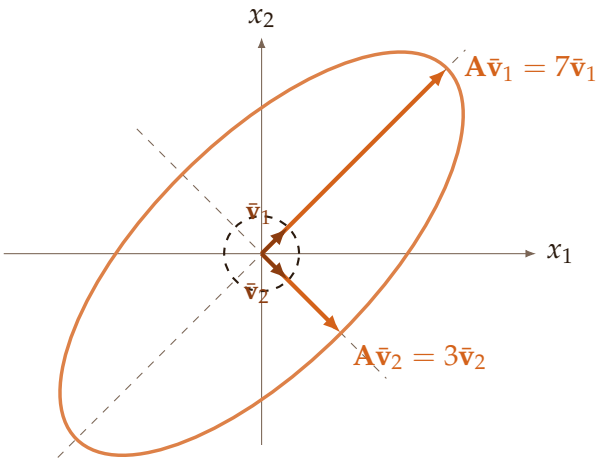


Figure 8.1 For $\mathbf{A} = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$ (Example 8.1), the unit circle (dashed) is mapped to an ellipse. The eigenvectors $\bar{\mathbf{v}}_1 \propto (1,1)$ and $\bar{\mathbf{v}}_2 \propto (1,-1)$ keep their directions and are stretched by $\lambda_1 = 7$ and $\lambda_2 = 3$.

Theorem 8.2 (Characteristic equation)

A scalar λ is an eigenvalue of $\mathbf{A}$ if and only if λ is a solution of the ***characteristic equation***

$$|\mathbf{A} - \lambda \mathbf{I}| = 0. \tag{8.2}$$

The corresponding eigenvectors are the nonzero solutions of $(\mathbf{A} - \lambda \mathbf{I})\bar{\mathbf{v}} = \mathbf{0}$, that is, the nonzero vectors in the null space of $\mathbf{A} - \lambda \mathbf{I}$.

8.2 Worked examples

Example 8.1 (A 2×2 matrix)

Determine the eigenvalues and eigenvectors of $\mathbf{A} = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$.

Solution. *Eigenvalues.*

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} = 0 \;\Rightarrow\; (5-\lambda)^2 - 4 = 0$$
$$\Rightarrow\; \lambda^2 - 10\lambda + 21 = 0$$
$$\Rightarrow\; (\lambda - 7)(\lambda - 3) = 0,$$

so $\lambda = 3, 7$.

Eigenvectors. They are found by solving the homogeneous system $(\mathbf{A} - \lambda \mathbf{I})\bar{\mathbf{v}} = \mathbf{0}$. For $\lambda = 7$,

$$(\mathbf{A} - 7\mathbf{I})\bar{\mathbf{v}} = \left(\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \right) \bar{\mathbf{v}} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathbf{0}.$$

Simplifying, $-2v_1 + 2v_2 = 0 \Rightarrow v_1 = v_2$, so

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For the other eigenvalue, $\lambda = 3$,

$$(\mathbf{A} - 3\mathbf{I})\bar{\mathbf{v}} = \left(\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right) \bar{\mathbf{v}} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathbf{0},$$

which simplifies to $2v_1 + 2v_2 = 0 \Rightarrow v_1 = -v_2$, so

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = k \begin{bmatrix} 1 \\ -1 \end{bmatrix} \sim \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Therefore the eigenvalues and eigenvectors are

$$\lambda_1 = 7,\ \bar{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \qquad \lambda_2 = 3,\ \bar{\mathbf{v}}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Here $\sim$ means equal up to a nonzero scale factor k: any nonzero multiple of an eigenvector is again an eigenvector. ◂

Example 8.2 (A 3×3 matrix with a repeated eigenvalue)

Find the eigenvalues and eigenvectors of $\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 \\ -1 & 0 & -1 \\ 2 & 4 & 4 \end{bmatrix}$.

Solution. *Eigenvalues.* The characteristic polynomial is

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 3-\lambda & 2 & 1 \\ -1 & -\lambda & -1 \\ 2 & 4 & 4-\lambda \end{vmatrix}.$$

Expanding along the first row,

$$\begin{aligned} |\mathbf{A} - \lambda\mathbf{I}| &= (3-\lambda)\big((-\lambda)(4-\lambda)+4\big) - 2(-4+\lambda+2) + 1(-4+2\lambda) \\ &= (3-\lambda)(\lambda^2 - 4\lambda + 4) - 2(\lambda-2) + 2(\lambda-2) \\ &= -\lambda^3 + 7\lambda^2 - 16\lambda + 12 = 0 \ \Rightarrow\ -(\lambda-2)^2(\lambda-3) = 0, \end{aligned}$$

so $\lambda = 2, 3$. There is a double eigenvalue $\lambda = 2$; the number of times an eigenvalue is repeated as a root is its **algebraic multiplicity** (here 2).
Eigenvectors for $\lambda = 2$.

$$(\mathbf{A} - 2\mathbf{I})\bar{\mathbf{v}} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0} \ \Rightarrow\ x + 2y + z = 0.$$

Therefore the eigenvectors are of the form

$$\begin{bmatrix} -2y - z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Eigenvector for $\lambda = 3$.

$$(\mathbf{A} - 3\mathbf{I})\bar{\mathbf{v}} = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -3 & -1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0} \;\Rightarrow\; 2y + z = 0, \quad x + 3y + z = 0.$$

Hence $z = -2y$ and $x = -y$, and the eigenvector is of the form

$$\begin{bmatrix} x \\ -x \\ 2x \end{bmatrix} = x \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

Therefore the matrix has 3 eigenvectors, which span $\mathbb{R}^3$:

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

◀

8.3 Complete eigenspace

Definition 8.3 (Complete eigenspace). The eigenspace of an $n \times n$ real matrix is **complete** if its eigenvectors span $\mathbb{R}^n$.

Theorem 8.4 (Independence of eigenvectors)

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct eigenvalues of a matrix $\mathbf{A}$. Then the corresponding eigenvectors are linearly independent.

Proof. We argue by induction. Suppose it is true for $k-1$ eigenvectors, that is, any set of $k-1$ eigenvectors is linearly independent. Suppose, to the contrary, that $\bar{\mathbf{v}}_{i_1}, \bar{\mathbf{v}}_{i_2}, \ldots, \bar{\mathbf{v}}_{i_k}$ are eigenvectors with scalars $c_{i_j} \neq 0$ such that

$$c_{i_1}\bar{\mathbf{v}}_{i_1} + \cdots + c_{i_k}\bar{\mathbf{v}}_{i_k} = \mathbf{0}.$$

Then, multiplying by $\mathbf{A}$,

$$\mathbf{A}\left(c_{i_1}\bar{\mathbf{v}}_{i_1} + \cdots + c_{i_k}\bar{\mathbf{v}}_{i_k}\right) = \mathbf{0} \;\Rightarrow\; c_{i_1}\lambda_{i_1}\bar{\mathbf{v}}_{i_1} + \cdots + c_{i_k}\lambda_{i_k}\bar{\mathbf{v}}_{i_k} = \mathbf{0}.$$

Further, multiplying the original equation by λ_{i_k},

$$c_{i_1}\lambda_{i_k}\bar{\mathbf{v}}_{i_1} + \cdots + c_{i_k}\lambda_{i_k}\bar{\mathbf{v}}_{i_k} = \mathbf{0}.$$

Subtracting one from the other, the last term cancels:

$$c_{i_1}(\lambda_{i_1} - \lambda_{i_k})\bar{\mathbf{v}}_{i_1} + \cdots + c_{i_{k-1}}(\lambda_{i_{k-1}} - \lambda_{i_k})\bar{\mathbf{v}}_{i_{k-1}} = \mathbf{0}.$$

Since the eigenvalues are distinct, every coefficient $c_{i_j}(\lambda_{i_j} - \lambda_{i_k})$ is nonzero. This contradicts the assumption that any set of $k-1$ eigenvectors is linearly independent. ∎

Therefore, suppose an $n \times n$ real matrix $\mathbf{A}$ has n distinct real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then the corresponding real eigenvectors $\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \ldots, \bar{\mathbf{v}}_n$ form a basis of $\mathbb{R}^n$; hence its eigenspace is complete.

> **Remark.** Distinct eigenvalues are sufficient, not necessary. The matrix of Example 8.2 has a repeated eigenvalue, yet its three eigenvectors still span $\mathbb{R}^3$.

8.4 The eigenvalue decomposition

Consider an $n \times n$ real matrix $\mathbf{A}$ with eigenvectors $\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \ldots, \bar{\mathbf{v}}_n$ that form a basis of $\mathbb{R}^n$:

$$\mathbf{A}\bar{\mathbf{v}}_1 = \lambda_1\bar{\mathbf{v}}_1, \qquad \mathbf{A}\bar{\mathbf{v}}_2 = \lambda_2\bar{\mathbf{v}}_2, \qquad \ldots, \qquad \mathbf{A}\bar{\mathbf{v}}_n = \lambda_n\bar{\mathbf{v}}_n.$$

Define the matrix of eigenvectors $\mathbf{V}$ and the diagonal matrix of eigenvalues Λ as

$$\mathbf{V} = \begin{bmatrix} | & | & & | \\ \bar{\mathbf{v}}_1 & \bar{\mathbf{v}}_2 & \cdots & \bar{\mathbf{v}}_n \\ | & | & & | \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}.$$

The n equations $\mathbf{A}\bar{\mathbf{v}}_i = \lambda_i\bar{\mathbf{v}}_i$ then combine into

$$\mathbf{A}\begin{bmatrix} | & | & & | \\ \bar{\mathbf{v}}_1 & \bar{\mathbf{v}}_2 & \cdots & \bar{\mathbf{v}}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \bar{\mathbf{v}}_1 & \bar{\mathbf{v}}_2 & \cdots & \bar{\mathbf{v}}_n \\ | & | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix},$$

that is, $\mathbf{AV} = \mathbf{V}\Lambda$. Since the columns of $\mathbf{V}$ form a basis, $\mathbf{V}$ is invertible.

Theorem 8.5 (Eigenvalue decomposition)

If $\mathbf{A}$ has n linearly independent eigenvectors, then

$$\mathbf{AV} = \mathbf{V}\Lambda \;\Rightarrow\; \mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^{-1}. \tag{8.3}$$

*This is the **eigenvalue decomposition** (EVD) of $\mathbf{A}$. Furthermore,*

$$\mathbf{V}^{-1}\mathbf{AV} = \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix},$$

*and $\mathbf{A}$ is said to be **diagonalizable**.*

Not all matrices are diagonalizable; we will explore this later.

Example 8.3 (An eigenvalue decomposition)

Find the eigenvalue decomposition of the matrix of Example 8.2, $\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 \\ -1 & 0 & -1 \\ 2 & 4 & 4 \end{bmatrix}$.

Solution. As seen earlier, the eigenvectors are $\begin{bmatrix} -2 & 1 & 0 \end{bmatrix}^\mathsf{T}$, $\begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^\mathsf{T}$ and $\begin{bmatrix} 1 & -1 & 2 \end{bmatrix}^\mathsf{T}$, for the eigenvalues 2, 2 and 3. The eigenvector matrix and its inverse are

$$\mathbf{V} = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}, \qquad \mathbf{V}^{-1} = \begin{bmatrix} 1 & 3 & 1 \\ -2 & -4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

The eigenvalue decomposition is

$$\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^{-1} = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ -2 & -4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

It can then be checked that

$$\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \begin{bmatrix} 1 & 3 & 1 \\ -2 & -4 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ -1 & 0 & -1 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \Lambda.$$

◀

8.5 Trace and determinant from the eigenvalues

Theorem 8.6 (Trace and determinant)

Let $\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^{-1}$ *be diagonalizable. Then*

$$\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i, \qquad \det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i. \tag{8.4}$$

Thus the trace of a matrix equals the sum of its eigenvalues, and the determinant equals the product of its eigenvalues.

Proof. Using $\operatorname{tr}(\mathbf{XY}) = \operatorname{tr}(\mathbf{YX})$,

$$\operatorname{tr}(\mathbf{A}) = \operatorname{tr}(\mathbf{V}\Lambda\mathbf{V}^{-1}) = \operatorname{tr}(\Lambda\mathbf{V}^{-1}\mathbf{V}) = \operatorname{tr}(\Lambda) = \sum_{i=1}^{n} \lambda_i.$$

Using $\det(\mathbf{XY}) = \det(\mathbf{X})\det(\mathbf{Y})$ and $\det(\mathbf{V}^{-1}) = 1/\det(\mathbf{V})$,

$$\det(\mathbf{A}) = \det(\mathbf{V}\Lambda\mathbf{V}^{-1}) = \det(\mathbf{V})\det(\Lambda)\det(\mathbf{V}^{-1}) = \det\Lambda = \prod_{i=1}^{n} \lambda_i.$$

■

Example 8.4 (Checking trace and determinant)

For the matrix of Example 8.2, $\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 \\ -1 & 0 & -1 \\ 2 & 4 & 4 \end{bmatrix}$, with eigenvalues $2, 2, 3$, verify Theorem 8.6.

Solution. *Trace.* $\operatorname{tr}(\mathbf{A}) = 3 + 0 + 4 = 7$, and $\sum_{i=1}^{3} \lambda_i = 2 + 2 + 3 = 7$.
Determinant. By the Laplace formula,

$$\det(\mathbf{A}) = 3(0+4) - 2(-4+2) + 1(-4-0) = 12 + 4 - 4 = 12,$$

which equals the product of the eigenvalues, $\prod_{i=1}^{3} \lambda_i = 2 \times 2 \times 3 = 12$. ◂

8.6 Symmetric matrices

Theorem 8.7 (Real eigenvalues)

The eigenvalues of a symmetric (Hermitian) matrix $\mathbf{A} = \mathbf{A}^{\mathsf{H}}$ are real, that is, $\lambda \in \mathbb{R}$.

Proof. Let λ be an eigenvalue with eigenvector $\bar{\mathbf{v}}$. Then $\mathbf{A}\bar{\mathbf{v}} = \lambda\bar{\mathbf{v}}$, so

$$\bar{\mathbf{v}}^{\mathsf{H}}\mathbf{A}\bar{\mathbf{v}} = \bar{\mathbf{v}}^{\mathsf{H}}\lambda\bar{\mathbf{v}} = \lambda\|\bar{\mathbf{v}}\|^2 \;\Rightarrow\; \lambda = \frac{\bar{\mathbf{v}}^{\mathsf{H}}\mathbf{A}\bar{\mathbf{v}}}{\|\bar{\mathbf{v}}\|^2}.$$

Further, since $\bar{\mathbf{v}}^{\mathsf{H}}\mathbf{A}\bar{\mathbf{v}}$ is a scalar,

$$\left(\bar{\mathbf{v}}^{\mathsf{H}}\mathbf{A}\bar{\mathbf{v}}\right)^{*} = \left(\bar{\mathbf{v}}^{\mathsf{H}}\mathbf{A}\bar{\mathbf{v}}\right)^{\mathsf{H}} = \bar{\mathbf{v}}^{\mathsf{H}}\mathbf{A}^{\mathsf{H}}\bar{\mathbf{v}} = \bar{\mathbf{v}}^{\mathsf{H}}\mathbf{A}\bar{\mathbf{v}} \;\Rightarrow\; \bar{\mathbf{v}}^{\mathsf{H}}\mathbf{A}\bar{\mathbf{v}} \in \mathbb{R}.$$

Therefore, since $\bar{\mathbf{v}}^{\mathsf{H}}\mathbf{A}\bar{\mathbf{v}}$ and $\|\bar{\mathbf{v}}\|^2$ are real,

$$\lambda = \frac{\bar{\mathbf{v}}^{\mathsf{H}}\mathbf{A}\bar{\mathbf{v}}}{\|\bar{\mathbf{v}}\|^2} \in \mathbb{R}.$$

■

Theorem 8.8 (Orthogonal eigenvectors)

For a symmetric (Hermitian) matrix $\mathbf{A}$, eigenvectors corresponding to distinct eigenvalues are orthogonal.

Proof. Let $\mathbf{A}\bar{\mathbf{u}} = \lambda\bar{\mathbf{u}}$ and $\mathbf{A}\bar{\mathbf{v}} = \mu\bar{\mathbf{v}}$ with $\lambda \neq \mu$. Then

$$\bar{\mathbf{v}}^{\mathsf{H}}\mathbf{A}\bar{\mathbf{u}} = \bar{\mathbf{v}}^{\mathsf{H}}\lambda\bar{\mathbf{u}}, \qquad \bar{\mathbf{u}}^{\mathsf{H}}\mathbf{A}\bar{\mathbf{v}} = \bar{\mathbf{u}}^{\mathsf{H}}\mu\bar{\mathbf{v}}.$$

Taking the Hermitian transpose of the second scalar and using $\mathbf{A}^{\mathsf{H}} = \mathbf{A}$,

$$\left(\bar{\mathbf{u}}^{\mathsf{H}}\mathbf{A}\bar{\mathbf{v}}\right)^{\mathsf{H}} = \bar{\mathbf{v}}^{\mathsf{H}}\mathbf{A}^{\mathsf{H}}\bar{\mathbf{u}} = \bar{\mathbf{v}}^{\mathsf{H}}\mathbf{A}\bar{\mathbf{u}} = \bar{\mathbf{v}}^{\mathsf{H}}\lambda\bar{\mathbf{u}}.$$

Furthermore, since $\mu \in \mathbb{R}$ by Theorem 8.7, we have $\mu = \mu^*$, so

$$\left(\bar{\mathbf{u}}^{\mathsf{H}}\mathbf{A}\bar{\mathbf{v}}\right)^{\mathsf{H}} = \left(\bar{\mathbf{u}}^{\mathsf{H}}\mu\bar{\mathbf{v}}\right)^{\mathsf{H}} = \bar{\mathbf{v}}^{\mathsf{H}}\mu^{*}\bar{\mathbf{u}} = \bar{\mathbf{v}}^{\mathsf{H}}\mu\bar{\mathbf{u}}.$$

Thus $\bar{\mathbf{v}}^{\mathsf{H}}\mu\bar{\mathbf{u}} = \bar{\mathbf{v}}^{\mathsf{H}}\lambda\bar{\mathbf{u}}$, that is,

$$(\lambda - \mu)\,\bar{\mathbf{v}}^{\mathsf{H}}\bar{\mathbf{u}} = 0.$$

Since $\lambda \neq \mu$, it follows that $\bar{\mathbf{v}}^{\mathsf{H}}\bar{\mathbf{u}} = 0$. ■

Theorem 8.9 (Spectral theorem)

For a symmetric $n \times n$ matrix $\mathbf{A}$, there is an orthonormal basis of $\mathbb{R}^n$ consisting of n eigenvectors of $\mathbf{A}$:

$$\bar{\mathbf{v}}_i^{\mathsf{H}}\bar{\mathbf{v}}_j = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Therefore the eigenvector matrix $\mathbf{V} = \begin{bmatrix} \bar{\mathbf{v}}_1 & \bar{\mathbf{v}}_2 & \cdots & \bar{\mathbf{v}}_n \end{bmatrix}$ satisfies $\mathbf{V}\mathbf{V}^{\mathsf{H}} = \mathbf{V}^{\mathsf{H}}\mathbf{V} = \mathbf{I}$, so $\mathbf{V}^{-1} = \mathbf{V}^{\mathsf{H}}$ and the eigenvalue decomposition is

$$\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^{-1} = \mathbf{V}\Lambda\mathbf{V}^{\mathsf{H}}. \tag{8.5}$$

For a real symmetric matrix $\mathbf{A} = \mathbf{A}^{\mathsf{T}}$, this reads $\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^{-1} = \mathbf{V}\Lambda\mathbf{V}^{\mathsf{T}}$.

Example 8.5 (EVD of a symmetric matrix)

Write the eigenvalue decomposition of $\mathbf{A} = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$ of Example 8.1 in the form $\mathbf{V}\Lambda\mathbf{V}^{\mathsf{T}}$.

Solution. The eigenvectors $\begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathsf{T}}$ and $\begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathsf{T}}$ are orthogonal, as Theorem 8.8 predicts. Normalizing them,

$$\mathbf{A} = \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{\mathbf{V}} \underbrace{\begin{bmatrix} 7 & 0 \\ 0 & 3 \end{bmatrix}}_{\Lambda} \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{\mathbf{V}^{\mathsf{T}}},$$

with $\mathbf{V}\mathbf{V}^{\mathsf{T}} = \mathbf{I}$. ◂

8.7 Positive definite and semidefinite matrices

Recall that a symmetric matrix $\mathbf{A} = \mathbf{A}^{\mathsf{T}}$ is positive semidefinite if $\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{A}\bar{\mathbf{x}} \geq 0$ for every $\bar{\mathbf{x}}$, and positive definite if $\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{A}\bar{\mathbf{x}} > 0$ for every $\bar{\mathbf{x}} \neq \mathbf{0}$.

Theorem 8.10 (Eigenvalues of PSD and PD matrices)

The eigenvalues of a positive semidefinite matrix are non-negative, and the eigenvalues of a positive definite matrix are positive.

Proof. Let $\mathbf{A}\bar{\mathbf{x}} = \lambda\bar{\mathbf{x}}$. Then

$$\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{A}\bar{\mathbf{x}} = \bar{\mathbf{x}}^{\mathsf{T}}\lambda\bar{\mathbf{x}} = \lambda\|\bar{\mathbf{x}}\|^2.$$

Since $\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{A}\bar{\mathbf{x}} \geq 0$ and $\|\bar{\mathbf{x}}\|^2 > 0$,

$$\lambda = \frac{\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{A}\bar{\mathbf{x}}}{\|\bar{\mathbf{x}}\|^2} \geq 0.$$

Similarly, since $\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{A}\bar{\mathbf{x}} > 0$ for a positive definite matrix,

$$\lambda = \frac{\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{A}\bar{\mathbf{x}}}{\|\bar{\mathbf{x}}\|^2} > 0.$$

■

Theorem 8.11 (Positive eigenvalues imply positive definite)

Conversely, if a symmetric matrix $\mathbf{A}$ *has positive eigenvalues* λ_i*, then it is positive definite.*

Proof. Consider orthonormal eigenvectors $\bar{\mathbf{v}}_i$ of $\mathbf{A}$ (Theorem 8.9),

$$\bar{\mathbf{v}}_i^{\mathsf{T}}\bar{\mathbf{v}}_j = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Any vector $\bar{\mathbf{x}}$ can be expressed as $\bar{\mathbf{x}} = c_1\bar{\mathbf{v}}_1 + c_2\bar{\mathbf{v}}_2 + \cdots + c_n\bar{\mathbf{v}}_n$. Therefore

$$\begin{aligned} \bar{\mathbf{x}}^{\mathsf{T}}\mathbf{A}\bar{\mathbf{x}} &= (c_1\bar{\mathbf{v}}_1 + c_2\bar{\mathbf{v}}_2 + \cdots + c_n\bar{\mathbf{v}}_n)^{\mathsf{T}}\,\mathbf{A}\,(c_1\bar{\mathbf{v}}_1 + c_2\bar{\mathbf{v}}_2 + \cdots + c_n\bar{\mathbf{v}}_n) \\ &= (c_1\bar{\mathbf{v}}_1 + c_2\bar{\mathbf{v}}_2 + \cdots + c_n\bar{\mathbf{v}}_n)^{\mathsf{T}}\,(c_1\lambda_1\bar{\mathbf{v}}_1 + c_2\lambda_2\bar{\mathbf{v}}_2 + \cdots + c_n\lambda_n\bar{\mathbf{v}}_n)\,. \end{aligned}$$

By orthonormality, this simplifies to

$$\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{A}\bar{\mathbf{x}} = \lambda_1 c_1^2 + \lambda_2 c_2^2 + \cdots + \lambda_n c_n^2.$$

For $\bar{\mathbf{x}} \neq \mathbf{0}$, not all c_i are zero, so the right-hand side is > 0 since every $\lambda_i > 0$. Therefore $\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{A}\bar{\mathbf{x}} > 0$ for any nonzero $\bar{\mathbf{x}}$, which implies that $\mathbf{A}$ is positive definite. ■

The same argument with $\lambda_i \geq 0$ shows that a symmetric matrix with non-negative eigenvalues is positive semidefinite. Hence a symmetric matrix is PSD (PD) if and only if all its eigenvalues are ≥ 0 (> 0).

Matrix	Eigenvalues	Eigenvectors / EVD
Diagonalizable	any	$\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^{-1}$
Symmetric / Hermitian	real	orthonormal, $\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^{\mathsf{H}}$
Positive semidefinite	real, $\lambda_i \geq 0$	orthonormal
Positive definite	real, $\lambda_i > 0$	orthonormal

Table 8.1 Eigenvalues and eigenvectors of special matrices.

AIML & 6G Connection

Principal component analysis is the EVD of a data covariance matrix, $\mathbf{C} = \mathbf{V}\Lambda\mathbf{V}^{\mathsf{T}}$: the eigenvectors are the principal directions and the eigenvalues the variances along them. In a MIMO link, the EVD of the Hermitian PSD matrix $\mathbf{H}^{\mathsf{H}}\mathbf{H} = \mathbf{V}\Lambda\mathbf{V}^{\mathsf{H}}$ splits the channel into independent *eigenmodes* with gains λ_i; transmitting along $\bar{\mathbf{v}}_i$ and allocating power by water-filling achieves capacity $\sum_i \log_2\left(1 + \lambda_i p_i/\sigma^2\right)$.

Chapter summary

- λ is an eigenvalue of $\mathbf{A}$ if $\mathbf{A}\bar{\mathbf{v}} = \lambda\bar{\mathbf{v}}$ for some $\bar{\mathbf{v}} \neq \mathbf{0}$, the eigenvector.
- Eigenvalues are the roots of the characteristic equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$.
- Eigenvectors for λ span the null space of $\mathbf{A} - \lambda\mathbf{I}$; a repeated root (algebraic multiplicity > 1) may give several.
- The eigenspace is complete if the eigenvectors span $\mathbb{R}^n$.
- Eigenvectors of distinct eigenvalues are linearly independent.
- With n independent eigenvectors, $\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^{-1}$ and $\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \Lambda$ is diagonal.
- The trace of a diagonalizable matrix equals the sum of its eigenvalues; the determinant equals their product.
- A symmetric (Hermitian) matrix $\mathbf{A} = \mathbf{A}^{\mathsf{H}}$ has real eigenvalues.
- Its eigenvectors for distinct eigenvalues are orthogonal, so $\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^{\mathsf{H}}$ with $\mathbf{V}\mathbf{V}^{\mathsf{H}} = \mathbf{I}$.
- A symmetric matrix is PSD (PD) if and only if all its eigenvalues are ≥ 0 (> 0).

Exercises

8.1 Find the eigenvalues and eigenvectors of $\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$, and write its eigenvalue decomposition.

8.2 Find an orthonormal eigenvector matrix $\mathbf{V}$ such that $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \mathbf{V}\Lambda\mathbf{V}^{\mathsf{T}}$.

8.3 Show that $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable.

8.4 Show that the eigenvalues of a triangular matrix are its diagonal entries.

8.5 If $\mathbf{A}\bar{\mathbf{v}} = \lambda\bar{\mathbf{v}}$, show that $\mathbf{A}^k\bar{\mathbf{v}} = \lambda^k\bar{\mathbf{v}}$, and that $\mathbf{A}^{-1}\bar{\mathbf{v}} = \lambda^{-1}\bar{\mathbf{v}}$ when $\mathbf{A}$ is invertible.

8.6 A 2×2 matrix has trace 7 and determinant 10. Find its eigenvalues.

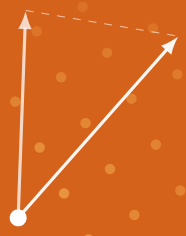
8.7 Show that $\mathbf{A}^{\mathsf{T}}\mathbf{A}$ is positive semidefinite for any real matrix $\mathbf{A}$, and hence that its eigenvalues are non-negative.

8.8 Verify Theorem 8.6 for $\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$.

09

Singular Value Decomposition

Singular value decomposition, rank and condition number, pseudoinverse, and the Jordan normal form



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9.1 The singular value decomposition

Theorem 9.1 (Singular value decomposition)

Any $m \times n$ matrix $\mathbf{A}$ has a ***singular value decomposition*** *(SVD)*

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^\mathsf{T} = \begin{bmatrix} | & | & & | \\ \bar{\mathbf{u}}_1 & \bar{\mathbf{u}}_2 & \cdots & \bar{\mathbf{u}}_m \\ | & | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \end{bmatrix} \begin{bmatrix} - & \bar{\mathbf{v}}_1^\mathsf{T} & - \\ - & \bar{\mathbf{v}}_2^\mathsf{T} & - \\ & \vdots & \\ - & \bar{\mathbf{v}}_n^\mathsf{T} & - \end{bmatrix}. \tag{9.1}$$

The factors have the following properties.

- $\mathbf{U}$ is of size $m \times m$ and $\mathbf{V}$ is of size $n \times n$. Both are orthogonal:

$$\mathbf{U}\mathbf{U}^\mathsf{T} = \mathbf{U}^\mathsf{T}\mathbf{U} = \mathbf{I}, \qquad \mathbf{V}\mathbf{V}^\mathsf{T} = \mathbf{V}^\mathsf{T}\mathbf{V} = \mathbf{I}.$$

- The columns $\bar{\mathbf{u}}_i$ of $\mathbf{U}$ therefore form an orthonormal set. They are the **left singular vectors**, and $\mathbf{U}$ is the **left singular matrix**.

- Similarly, the columns $\bar{\mathbf{v}}_i$ of $\mathbf{V}$ (the rows $\bar{\mathbf{v}}_i^{\mathsf{T}}$ of $\mathbf{V}^{\mathsf{T}}$) form an orthonormal set. They are the **right singular vectors**, and $\mathbf{V}$ is the **right singular matrix**.
- Σ is of size $m \times n$. It has the **singular values** σ_i, $i = 1, 2, \ldots, l$, along its principal diagonal, where $l = \min\{m, n\}$, and zeros elsewhere.
- The singular values are non-negative and arranged in decreasing order, $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq 0$.

Remark. For a complex matrix the same holds with unitary $\mathbf{U}$ and $\mathbf{V}$: $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathsf{H}}$.

9.1.1 Singular vectors and singular values

Theorem 9.2 (Action on the right singular vectors)

For each i,

$$\mathbf{A}\bar{\mathbf{v}}_i = \sigma_i \bar{\mathbf{u}}_i. \tag{9.2}$$

Proof. Consider $i = 1$. Since the rows of $\mathbf{V}^{\mathsf{T}}$ are orthonormal,

$$\mathbf{A}\bar{\mathbf{v}}_1 = \mathbf{U}\Sigma\mathbf{V}^{\mathsf{T}}\bar{\mathbf{v}}_1 = \mathbf{U}\Sigma \begin{bmatrix} - & \bar{\mathbf{v}}_1^{\mathsf{T}} & - \\ - & \bar{\mathbf{v}}_2^{\mathsf{T}} & - \\ & \vdots & \\ - & \bar{\mathbf{v}}_n^{\mathsf{T}} & - \end{bmatrix} \bar{\mathbf{v}}_1 = \mathbf{U}\Sigma \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

This simplifies to

$$\mathbf{A}\bar{\mathbf{v}}_1 = \begin{bmatrix} | & | & & | \\ \bar{\mathbf{u}}_1 & \bar{\mathbf{u}}_2 & \cdots & \bar{\mathbf{u}}_m \\ | & | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

which reduces to

$$\mathbf{A}\bar{\mathbf{v}}_1 = \begin{bmatrix} | & | & & | \\ \bar{\mathbf{u}}_1 & \bar{\mathbf{u}}_2 & \cdots & \bar{\mathbf{u}}_m \\ | & | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \sigma_1 \bar{\mathbf{u}}_1.$$

Generalizing, it is easy to see that $\mathbf{A}\bar{\mathbf{v}}_i = \sigma_i \bar{\mathbf{u}}_i$. ■

Geometrically, $\mathbf{V}^{\mathsf{T}}$ rotates, Σ stretches along the axes, and $\mathbf{U}$ rotates again: $\mathbf{A}$ maps the unit circle to an ellipse whose semi-axes are $\sigma_i \bar{\mathbf{u}}_i$ (Figure 9.1).

Theorem 9.3 (Singular values from $\mathbf{A}^{\mathsf{T}}\mathbf{A}$ and $\mathbf{A}\mathbf{A}^{\mathsf{T}}$)

The right singular vectors $\bar{\mathbf{v}}_i$ are eigenvectors of $\mathbf{A}^{\mathsf{T}}\mathbf{A}$, the left singular vectors $\bar{\mathbf{u}}_i$ are eigenvectors of $\mathbf{A}\mathbf{A}^{\mathsf{T}}$, and in both cases the eigenvalues are $\lambda_i = \sigma_i^2$. In other words,

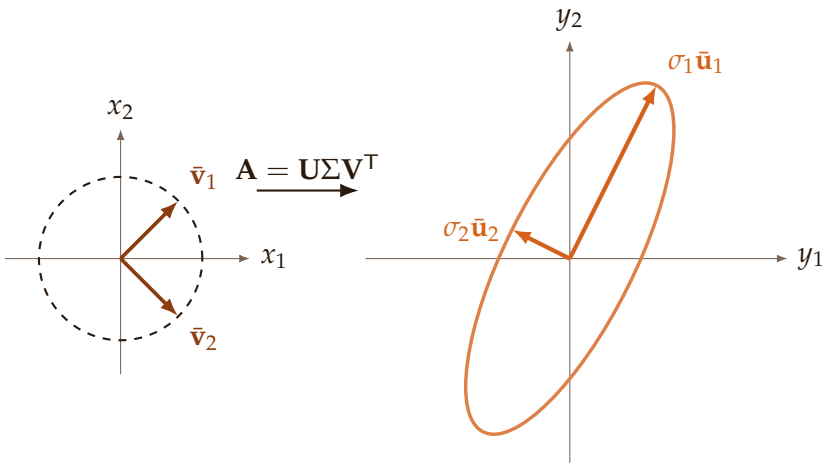


Figure 9.1 For $\mathbf{A} = \begin{bmatrix} 1 & 5 \\ 7 & 5 \end{bmatrix}$ (Example 9.1), the unit circle is mapped to an ellipse with semi-axes $\sigma_1\bar{\mathbf{u}}_1$ and $\sigma_2\bar{\mathbf{u}}_2$, where $\sigma_1 = \sqrt{90}$ and $\sigma_2 = \sqrt{10}$. The two panels are drawn at different scales.

$$\sigma_i = \sqrt{\lambda_i}, \qquad \lambda_i \text{ an eigenvalue of } \mathbf{A}^\mathsf{T}\mathbf{A}. \tag{9.3}$$

Proof. From $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^\mathsf{T}$ and $\mathbf{U}^\mathsf{T}\mathbf{U} = \mathbf{I}$,

$$\mathbf{A}^\mathsf{T}\mathbf{A} = \mathbf{V}\Sigma^\mathsf{T}\Sigma\mathbf{V}^\mathsf{T}.$$

Here $\Sigma^\mathsf{T}\Sigma$ is diagonal:

$$\Sigma^\mathsf{T}\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \end{bmatrix}^\mathsf{T} \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \ddots \end{bmatrix}.$$

So $\mathbf{A}^\mathsf{T}\mathbf{A} = \mathbf{V}(\Sigma^\mathsf{T}\Sigma)\mathbf{V}^\mathsf{T}$ is an eigenvalue decomposition (Chapter 8): the $\bar{\mathbf{v}}_i$ are eigenvectors of $\mathbf{A}^\mathsf{T}\mathbf{A}$ with eigenvalues $\lambda_i = \sigma_i^2$. Similarly,

$$\mathbf{A}\mathbf{A}^\mathsf{T} = \mathbf{U}\Sigma\mathbf{V}^\mathsf{T}\mathbf{V}\Sigma^\mathsf{T}\mathbf{U}^\mathsf{T} = \mathbf{U}\Sigma\Sigma^\mathsf{T}\mathbf{U}^\mathsf{T},$$

so the $\bar{\mathbf{u}}_i$ are eigenvectors of $\mathbf{A}\mathbf{A}^\mathsf{T}$, again with eigenvalues σ_i^2. ∎

9.2 Computing an SVD

Theorems 9.2 and 9.3 give a recipe: find the eigenvalues and orthonormal eigenvectors $\bar{\mathbf{v}}_i$ of $\mathbf{A}^\mathsf{T}\mathbf{A}$, take $\sigma_i = \sqrt{\lambda_i}$, and compute $\bar{\mathbf{u}}_i = \frac{1}{\sigma_i}\mathbf{A}\bar{\mathbf{v}}_i$.

Example 9.1 (SVD of a 2×2 matrix)

Find the SVD of $\mathbf{A} = \begin{bmatrix} 1 & 5 \\ 7 & 5 \end{bmatrix}$.

Solution. First,

$$\mathbf{A}^\top\mathbf{A} = \begin{bmatrix} 50 & 40 \\ 40 & 50 \end{bmatrix}.$$

Singular values. The eigenvalues of $\mathbf{A}^\top\mathbf{A}$ satisfy

$$|\mathbf{A}^\top\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 50-\lambda & 40 \\ 40 & 50-\lambda \end{vmatrix} = 0 \;\Rightarrow\; (50-\lambda)^2 = 40^2$$
$$\Rightarrow\; 50-\lambda = \pm 40 \;\Rightarrow\; \lambda = 90, 10,$$

so $\sigma_1 = \sqrt{90}$ and $\sigma_2 = \sqrt{10}$.
Right singular vectors. For $\lambda = 90$,

$$\mathbf{A}^\top\mathbf{A} - 90\mathbf{I} = \begin{bmatrix} -40 & 40 \\ 40 & -40 \end{bmatrix}, \qquad \begin{bmatrix} -40 & 40 \\ 40 & -40 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathbf{0} \;\Rightarrow\; v_1 = v_2,$$

so the unit eigenvector is $\bar{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sim \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. For $\lambda = 10$,

$$\mathbf{A}^\top\mathbf{A} - 10\mathbf{I} = \begin{bmatrix} 40 & 40 \\ 40 & 40 \end{bmatrix}, \qquad \begin{bmatrix} 40 & 40 \\ 40 & 40 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathbf{0} \;\Rightarrow\; v_1 + v_2 = 0 \;\Rightarrow\; v_1 = -v_2,$$

so $\bar{\mathbf{v}}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \sim \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$.
Left singular vectors. From $\mathbf{A}\bar{\mathbf{v}}_1 = \sigma_1\bar{\mathbf{u}}_1$, $\bar{\mathbf{u}}_1 = \frac{1}{\sigma_1}\mathbf{A}\bar{\mathbf{v}}_1$:

$$\bar{\mathbf{u}}_1 = \frac{1}{\sqrt{90}}\begin{bmatrix} 1 & 5 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{180}}\begin{bmatrix} 6 \\ 12 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}.$$

Next, from $\mathbf{A}\bar{\mathbf{v}}_2 = \sigma_2\bar{\mathbf{u}}_2$, $\bar{\mathbf{u}}_2 = \frac{1}{\sigma_2}\mathbf{A}\bar{\mathbf{v}}_2$:

$$\bar{\mathbf{u}}_2 = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & 5 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{20}}\begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}.$$

Therefore the SVD of $\mathbf{A}$ is

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^\top = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} \sqrt{90} & 0 \\ 0 & \sqrt{10} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

◀

9.3 SVD and rank

Theorem 9.4 (Rank and the four fundamental subspaces)

Let $\sigma_1, \sigma_2, \ldots, \sigma_r > 0$ *and* $\sigma_{r+1} = \sigma_{r+2} = \cdots = 0$. *Then*

$$r = \operatorname{rank}(\mathbf{A}), \tag{9.4}$$

that is, the rank of a matrix equals its number of nonzero singular values. Moreover:

- $\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \ldots, \bar{\mathbf{u}}_r$ *form an orthonormal basis of* $\operatorname{colsp}(\mathbf{A})$;
- $\bar{\mathbf{u}}_{r+1}, \bar{\mathbf{u}}_{r+2}, \ldots, \bar{\mathbf{u}}_m$ *form an orthonormal basis of* $\mathcal{N}(\mathbf{A}^\mathsf{T})$;
- $\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \ldots, \bar{\mathbf{v}}_r$ *form an orthonormal basis of* $\operatorname{rowsp}(\mathbf{A})$;
- $\bar{\mathbf{v}}_{r+1}, \bar{\mathbf{v}}_{r+2}, \ldots, \bar{\mathbf{v}}_n$ *form an orthonormal basis of* $\mathcal{N}(\mathbf{A})$.

Thus the SVD gives orthonormal bases of all four fundamental subspaces of any matrix $\mathbf{A}$ (Chapter 6).

Subspace	Dimension	Orthonormal basis from the SVD
Column space $\operatorname{colsp}(\mathbf{A})$	r	$\bar{\mathbf{u}}_1, \ldots, \bar{\mathbf{u}}_r$
Null space of $\mathbf{A}^\mathsf{T}$	$m - r$	$\bar{\mathbf{u}}_{r+1}, \ldots, \bar{\mathbf{u}}_m$
Row space $\operatorname{rowsp}(\mathbf{A})$	r	$\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_r$
Null space $\mathcal{N}(\mathbf{A})$	$n - r$	$\bar{\mathbf{v}}_{r+1}, \ldots, \bar{\mathbf{v}}_n$

Table 9.1 The four fundamental subspaces of an $m \times n$ matrix of rank r, from its SVD.

9.4 Condition number

Definition 9.5 (Condition number). The **condition number** of an $n \times n$ matrix is the ratio of its largest to its smallest singular value,

$$\kappa = \frac{\sigma_1}{\sigma_l}. \tag{9.5}$$

A matrix with a very large condition number is said to be **ill-conditioned**. In practice this occurs when the condition number is larger than the reciprocal of the machine's precision, e.g., 10^7 for most single-precision arithmetic. In the extreme case when $\mathbf{A}$ has one or more zero singular values, $\sigma_l = 0$, its condition number is infinite, and the linear system is singular, with either no solution or infinitely many solutions.

Example 9.2 (An ill-conditioned matrix)

Solution. Consider

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -4 & 4 \\ -1 & 2 & -2 \end{bmatrix}.$$

This matrix has rank $r = 1$. Now consider

$$\tilde{\mathbf{A}} = \begin{bmatrix} 1.00001 & -2 & 2 \\ 2 & -4.00001 & 4 \\ -1 & 2 & -2.00001 \end{bmatrix}.$$

Note that $\mathbf{A} \approx \tilde{\mathbf{A}}$. However, $\tilde{\mathbf{A}}$ has rank $r = 3$. Its singular values are

$$\sigma_1 = 7.34848, \qquad \sigma_2 \approx \sigma_3 \approx 10^{-5},$$

so

$$\kappa = \frac{\sigma_1}{\sigma_3} \approx 7.7 \times 10^5.$$

The condition number is very high. Hence, although its rank is 3, one should treat $\tilde{\mathbf{A}}$ as a rank-1 matrix. ◂

Definition 9.6 (Practical rank). A practical method of computing the rank of a matrix is to first assign a threshold for the singular values, e.g., 10^{-5}, and then treat any singular value lying below the threshold as if it were zero.

9.5 The pseudoinverse

Definition 9.7 (Pseudoinverse of Σ). Consider an $m \times n$ matrix Σ with nonzero elements $\sigma_1, \ldots, \sigma_r$ only along its principal diagonal. Such a matrix has no inverse if $m \neq n$. Its **pseudoinverse** is the $n \times m$ diagonal matrix $\Sigma^\dagger$ whose principal diagonal entries are the reciprocals of the nonzero singular values, $\frac{1}{\sigma_1}, \ldots, \frac{1}{\sigma_r}$, with zeros elsewhere:

$$\Sigma^\dagger = \begin{bmatrix} \frac{1}{\sigma_1} & & & \\ & \ddots & & \\ & & \frac{1}{\sigma_r} & \\ & & & 0 \end{bmatrix}.$$

Example 9.3 (Pseudoinverse of Σ)

For the 3×4 matrix

$$\Sigma = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

the pseudoinverse is the 4×3 matrix

$$\Sigma^\dagger = \begin{bmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Theorem 9.8 ($\Sigma^\dagger$ as a one-sided inverse)

If $m \geq n = r$, then $\Sigma^\dagger \Sigma = \mathbf{I}$. Similarly, if $n \geq m = r$, then $\Sigma\Sigma^\dagger = \mathbf{I}$.

Proof. For $m \geq n = r$, Σ has n nonzero diagonal entries above $m - n$ zero rows, so

$$\Sigma^\dagger \Sigma = \begin{bmatrix} \frac{1}{\sigma_1} & & & 0 \\ & \ddots & & \vdots \\ & & \frac{1}{\sigma_n} & 0 \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ 0 & \cdots & 0 \end{bmatrix} = \mathbf{I}.$$

The case $n \geq m = r$ is similar. ∎

Definition 9.9 (Pseudoinverse of a matrix). The **pseudoinverse** of an $m \times n$ matrix $\mathbf{A}$ with SVD $\mathbf{U}\Sigma\mathbf{V}^\mathsf{T}$ is

$$\mathbf{A}^\dagger = \mathbf{V}\Sigma^\dagger \mathbf{U}^\mathsf{T}. \tag{9.6}$$

9.5.1 Tall matrices: a left inverse

Theorem 9.10 (Tall, full column rank)

Let $\mathbf{A}$ be a tall $m \times n$ matrix with $m > n = \operatorname{rank}(\mathbf{A})$. Then $\mathbf{A}^\dagger$ is a left inverse of $\mathbf{A}$, $\mathbf{A}^\dagger\mathbf{A} = \mathbf{I}$, and

$$\mathbf{A}^\dagger = \left(\mathbf{A}^\mathsf{T}\mathbf{A}\right)^{-1}\mathbf{A}^\mathsf{T}. \tag{9.7}$$

Proof. Using $\mathbf{U}^\mathsf{T}\mathbf{U} = \mathbf{I}$ and Theorem 9.8,

$$\mathbf{A}^\dagger\mathbf{A} = \mathbf{V}\Sigma^\dagger\mathbf{U}^\mathsf{T}\mathbf{U}\Sigma\mathbf{V}^\mathsf{T} = \mathbf{V}\Sigma^\dagger\Sigma\mathbf{V}^\mathsf{T} = \mathbf{V}\mathbf{I}\mathbf{V}^\mathsf{T} = \mathbf{V}\mathbf{V}^\mathsf{T} = \mathbf{I}.$$

For (9.7), note that

$$\mathbf{A}^\mathsf{T}\mathbf{A} = \mathbf{V}\Sigma^\mathsf{T}\mathbf{U}^\mathsf{T}\mathbf{U}\Sigma\mathbf{V}^\mathsf{T} = \mathbf{V}\Sigma^\mathsf{T}\Sigma\mathbf{V}^\mathsf{T} = \mathbf{V}\tilde{\Sigma}^2\mathbf{V}^\mathsf{T},$$

where $\tilde{\Sigma}^2 = \Sigma^\mathsf{T}\Sigma$ is, in this case, an $n \times n$ diagonal matrix with entries σ_i^2, $i = 1, 2, \ldots, n$. Noting that $\tilde{\Sigma}^{-2}\Sigma^\mathsf{T} = \Sigma^\dagger$,

$$\left(\mathbf{A}^\mathsf{T}\mathbf{A}\right)^{-1}\mathbf{A}^\mathsf{T} = \mathbf{V}\tilde{\Sigma}^{-2}\mathbf{V}^\mathsf{T}\mathbf{V}\Sigma^\mathsf{T}\mathbf{U}^\mathsf{T} = \mathbf{V}\tilde{\Sigma}^{-2}\Sigma^\mathsf{T}\mathbf{U}^\mathsf{T} = \mathbf{V}\Sigma^\dagger\mathbf{U}^\mathsf{T} = \mathbf{A}^\dagger.$$

∎

9.5.2 Wide matrices: a right inverse

Theorem 9.11 (Wide, full row rank)

Let $\mathbf{A}$ *be a wide* $m \times n$ *matrix with* $n > m = \operatorname{rank}(\mathbf{A})$. *Then* $\mathbf{A}^\dagger$ *is a right inverse of* $\mathbf{A}$, $\mathbf{A}\mathbf{A}^\dagger = \mathbf{I}$, *and*

$$\mathbf{A}^\dagger = \mathbf{A}^\mathsf{T}\left(\mathbf{A}\mathbf{A}^\mathsf{T}\right)^{-1}. \tag{9.8}$$

Proof. Using $\mathbf{V}^\mathsf{T}\mathbf{V} = \mathbf{I}$ and Theorem 9.8,

$$\mathbf{A}\mathbf{A}^\dagger = \mathbf{U}\Sigma\mathbf{V}^\mathsf{T}\mathbf{V}\Sigma^\dagger\mathbf{U}^\mathsf{T} = \mathbf{U}\Sigma\Sigma^\dagger\mathbf{U}^\mathsf{T} = \mathbf{U}\mathbf{I}\mathbf{U}^\mathsf{T} = \mathbf{U}\mathbf{U}^\mathsf{T} = \mathbf{I}.$$

For (9.8), with $\tilde{\Sigma}^2 = \Sigma\Sigma^\mathsf{T}$ now an $m \times m$ diagonal matrix,

$$\begin{aligned}\mathbf{A}^\mathsf{T}\left(\mathbf{A}\mathbf{A}^\mathsf{T}\right)^{-1} &= \mathbf{V}\Sigma^\mathsf{T}\mathbf{U}^\mathsf{T}\left(\mathbf{U}\Sigma\mathbf{V}^\mathsf{T}\mathbf{V}\Sigma^\mathsf{T}\mathbf{U}^\mathsf{T}\right)^{-1} = \mathbf{V}\Sigma^\mathsf{T}\mathbf{U}^\mathsf{T}\left(\mathbf{U}\tilde{\Sigma}^2\mathbf{U}^\mathsf{T}\right)^{-1} \\ &= \mathbf{V}\Sigma^\mathsf{T}\mathbf{U}^\mathsf{T}\mathbf{U}\tilde{\Sigma}^{-2}\mathbf{U}^\mathsf{T} = \mathbf{V}\Sigma^\mathsf{T}\tilde{\Sigma}^{-2}\mathbf{U}^\mathsf{T} = \mathbf{V}\Sigma^\dagger\mathbf{U}^\mathsf{T} = \mathbf{A}^\dagger.\end{aligned}$$

■

AIML & 6G Connection

The tall-matrix pseudoinverse $(\mathbf{A}^\mathsf{T}\mathbf{A})^{-1}\mathbf{A}^\mathsf{T}$ is exactly the least-squares solution of the normal equations, used in linear regression. In MIMO, it is the *zero-forcing* receiver $\hat{\mathbf{x}} = \mathbf{H}^\dagger\bar{\mathbf{y}}$. The SVD itself splits a MIMO channel $\mathbf{H} = \mathbf{U}\Sigma\mathbf{V}^\mathsf{H}$ into parallel channels: precoding with $\mathbf{V}$ and combining with $\mathbf{U}^\mathsf{H}$ gives independent streams with gains σ_i. Truncating the SVD to its largest k singular values gives the best rank-k approximation of a matrix, used in recommender systems, image compression and low-rank adaptation of large language models.

9.6 Incomplete eigenspaces

Chapter 8 noted that not every matrix is diagonalizable. The next example shows why.

Example 9.4 (An incomplete eigenspace)

Consider $\mathbf{A} = \begin{bmatrix}1 & -1 & 3\\ 2 & 0 & 2\\ 2 & -1 & 2\end{bmatrix}$.

Solution. Expanding along the first row, the characteristic equation is

$$\begin{aligned}|\mathbf{A} - \lambda\mathbf{I}| &= \begin{vmatrix}1-\lambda & -1 & 3\\ 2 & -\lambda & 2\\ 2 & -1 & 2-\lambda\end{vmatrix} \\ &= (1-\lambda)\big((-\lambda)(2-\lambda)+2\big) + (4-2\lambda-4) + 3(-2+2\lambda) \\ &= (1-\lambda)(\lambda^2 - 2\lambda + 2) + 4\lambda - 6 = -\lambda^3 + 3\lambda^2 - 4 = 0.\end{aligned}$$

This factors as

$$-\lambda^3 + 3\lambda^2 - 4 = 0 \;\Rightarrow\; -(\lambda-2)^2(\lambda+1) = 0,$$

so there is a double eigenvalue at 2 and a simple eigenvalue at -1.
Consider

$$\mathbf{A} - \lambda_1 \mathbf{I} = \mathbf{A} - 2\mathbf{I} = \begin{bmatrix} -1 & -1 & 3 \\ 2 & -2 & 2 \\ 2 & -1 & 0 \end{bmatrix}.$$

To find the eigenvector, set

$$(\mathbf{A} - 2\mathbf{I})\bar{\mathbf{v}} = \begin{bmatrix} -1 & -1 & 3 \\ 2 & -2 & 2 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \mathbf{0}.$$

This simplifies to

$$2v_1 - v_2 = 0 \;\Rightarrow\; v_2 = 2v_1, \qquad -v_1 - v_2 + 3v_3 = 0 \;\Rightarrow\; 3v_3 = 3v_1 \;\Rightarrow\; v_3 = v_1.$$

Therefore the eigenvector is

$$\begin{bmatrix} v_1 \\ 2v_1 \\ v_1 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Corresponding to $\lambda_1 = 2$ there is only a *one-dimensional* eigenspace, although the algebraic multiplicity is 2. With one more eigenvector for $\lambda = -1$, there are only two independent eigenvectors. The eigenspace is therefore **incomplete**: the eigenvectors do not span $\mathbb{R}^3$, and $\mathbf{A}$ is not diagonalizable. ◂

9.7 The Jordan normal form

Definition 9.12 (Jordan block). A **Jordan block** is the bidiagonal matrix

$$\mathbf{J}_\lambda = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}.$$

Theorem 9.13 (Eigenvalues and eigenvectors of a Jordan block)

An $n \times n$ Jordan block $\mathbf{J}_\lambda$ has the single eigenvalue λ and, up to scaling, a single eigenvector $\bar{\mathbf{e}}_1$.

Proof. Observe that

$$\mathbf{J}_\lambda - \mu\mathbf{I} = \begin{bmatrix} \lambda - \mu & 1 & & \\ & \lambda - \mu & \ddots & \\ & & \ddots & 1 \\ & & & \lambda - \mu \end{bmatrix}.$$

To determine the eigenvalues, set the characteristic polynomial to zero. The matrix is triangular, so

$$|\mathbf{J}_\lambda - \mu\mathbf{I}| = (\lambda - \mu)^n = 0 \;\Rightarrow\; \mu = \lambda.$$

For $\mu = \lambda$, the equation $(\mathbf{J}_\lambda - \lambda\mathbf{I})\bar{\mathbf{v}} = \mathbf{0}$ reads $v_2 = 0, v_3 = 0, \ldots, v_n = 0$. Therefore the eigenvector is

$$\bar{\mathbf{v}} = k \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

so the block has only a single eigenvector. ■

A matrix $\mathbf{A}$ is diagonalizable if and only if it has n linearly independent eigenvectors that span $\mathbb{C}^n$. Matrices that are not diagonalizable can still be brought to a nearly diagonal form.

Theorem 9.14 (Jordan normal form)

Any square matrix $\mathbf{A}$ *can be reduced to a* ***Jordan normal form****:*

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{J} \;\Rightarrow\; \mathbf{A} = \mathbf{P}\mathbf{J}\mathbf{P}^{-1}, \tag{9.9}$$

where $\mathbf{J}$ *is the block diagonal matrix*

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_1 & & \\ & \ddots & \\ & & \mathbf{J}_p \end{bmatrix},$$

and each $\mathbf{J}_i$ *is a Jordan block*

$$\mathbf{J}_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}.$$

A diagonalizable matrix is the special case in which every Jordan block is 1×1.

Example 9.5 (Jordan normal form)

Determine the Jordan normal form of $\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & 2 & -1 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & 1 & -1 & 1 \end{bmatrix}$.

Solution. The characteristic polynomial is

$$\lambda^4 - 8\lambda^3 + 23\lambda^2 - 28\lambda + 12 = (\lambda - 1)(\lambda - 3)(\lambda - 2)^2 = 0 \;\Rightarrow\; \lambda = 1, 3, 2.$$

Eigenvectors corresponding to the eigenvalues 1 and 3 are

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}.$$

The eigenvalue 2 has multiplicity 2, but only one eigenvector, $\begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}^\mathsf{T}$. Therefore the eigenspace is incomplete.
For the eigenvalue 2, compute the kernel (null space) of $(\mathbf{A} - 2\mathbf{I})^2$:

$$\mathcal{N}\left((\mathbf{A} - 2\mathbf{I})^2\right) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

These are called **generalized eigenvectors**. The Jordan canonical form is

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{J} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \qquad \text{with} \qquad \mathbf{P} = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

The first three columns of $\mathbf{P}$ are the eigenvectors for 1, 3 and 2. The last column $\bar{\mathbf{p}}_4$ is a generalized eigenvector satisfying $(\mathbf{A} - 2\mathbf{I})\bar{\mathbf{p}}_4 = \bar{\mathbf{p}}_3$, which produces the 1 above the diagonal in the 2×2 Jordan block. ◂

Chapter summary

- Every matrix has an SVD $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^\mathsf{T}$ with orthogonal $\mathbf{U}$, $\mathbf{V}$ and diagonal Σ.
- Singular values satisfy $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$, and $\mathbf{A}\bar{\mathbf{v}}_i = \sigma_i \bar{\mathbf{u}}_i$.
- $\bar{\mathbf{v}}_i$ and $\bar{\mathbf{u}}_i$ are eigenvectors of $\mathbf{A}^\mathsf{T}\mathbf{A}$ and $\mathbf{A}\mathbf{A}^\mathsf{T}$, with eigenvalues σ_i^2.
- The rank is the number of nonzero singular values; $\mathbf{U}$ and $\mathbf{V}$ give orthonormal bases of the four fundamental subspaces.
- The condition number $\kappa = \sigma_1/\sigma_l$ is large for nearly singular matrices; in practice, singular values below a threshold are treated as zero.
- The pseudoinverse $\Sigma^\dagger$ inverts the nonzero singular values; $\mathbf{A}^\dagger = \mathbf{V}\Sigma^\dagger\mathbf{U}^\mathsf{T}$.
- Tall, full column rank: $\mathbf{A}^\dagger = (\mathbf{A}^\mathsf{T}\mathbf{A})^{-1}\mathbf{A}^\mathsf{T}$ is a left inverse.
- Wide, full row rank: $\mathbf{A}^\dagger = \mathbf{A}^\mathsf{T}(\mathbf{A}\mathbf{A}^\mathsf{T})^{-1}$ is a right inverse.
- A repeated eigenvalue can have too few eigenvectors: the eigenspace is incomplete.

- Every matrix has a Jordan normal form $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{J}$ built from Jordan blocks.

Exercises

9.1 Find the SVD of $\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$.

9.2 Write the matrix of Example 9.1 as a sum of rank-one matrices $\sigma_1 \bar{\mathbf{u}}_1 \bar{\mathbf{v}}_1^{\mathsf{T}} + \sigma_2 \bar{\mathbf{u}}_2 \bar{\mathbf{v}}_2^{\mathsf{T}}$, and find its best rank-one approximation.

9.3 Find the condition number of $\begin{bmatrix} 10 & 0 \\ 0 & 0.01 \end{bmatrix}$.

9.4 Find the pseudoinverses of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \end{bmatrix}$, and verify that they are a left and a right inverse.

9.5 Show that $\mathbf{A}\mathbf{A}^{\dagger}\mathbf{A} = \mathbf{A}$ and $\mathbf{A}^{\dagger}\mathbf{A}\mathbf{A}^{\dagger} = \mathbf{A}^{\dagger}$ for any matrix $\mathbf{A}$.

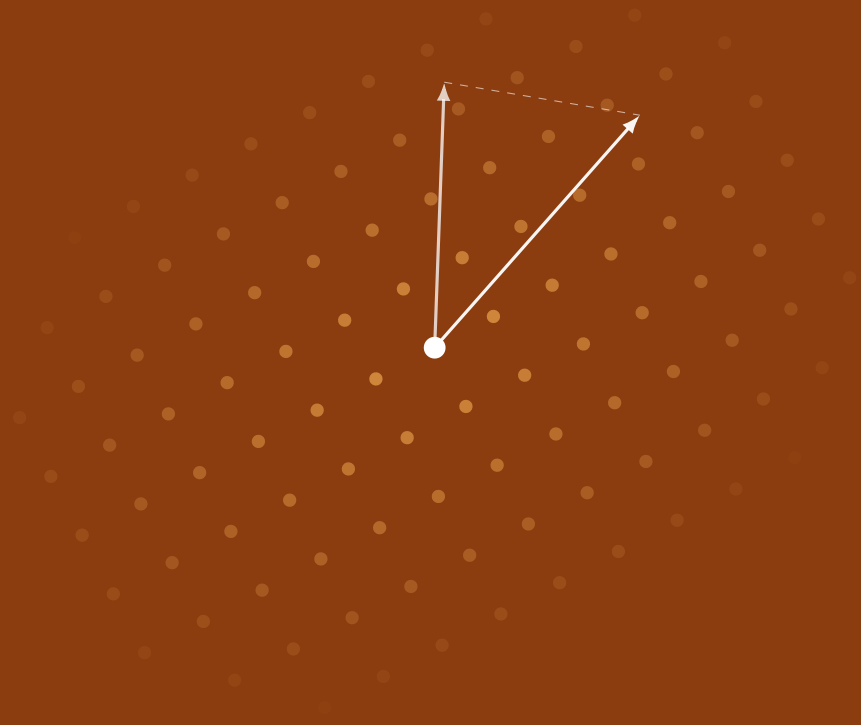
9.6 Show that for a square invertible matrix, $\mathbf{A}^{\dagger} = \mathbf{A}^{-1}$.

9.7 Find the Jordan normal form of $\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$.

9.8 Show that the singular values of a symmetric positive semidefinite matrix equal its eigenvalues.

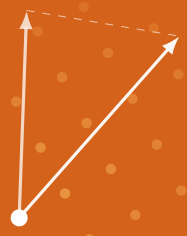
PART VI

Matrix Calculus, Least Square and Least Norm



10

Vector and Matrix Calculus



Vector and matrix derivatives, the Jacobian, the chain rule, and derivatives of the trace and determinant

In this chapter

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10.1 Vector derivatives

Throughout, $\bar{\mathbf{x}}$ denotes the vector

$$\bar{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Definition 10.1 (Derivative of a scalar with respect to a vector)**.** The derivative of a scalar function $y = y(\bar{\mathbf{x}})$ with respect to the vector $\bar{\mathbf{x}}$ is the column vector of partial derivatives

$$\frac{\partial y}{\partial \bar{\mathbf{x}}} = \begin{bmatrix} \frac{\partial y}{\partial x_1} \\ \frac{\partial y}{\partial x_2} \\ \vdots \\ \frac{\partial y}{\partial x_n} \end{bmatrix}. \tag{10.1}$$

It is also called the **gradient** and denoted ∇y.

Definition 10.2 (Derivative of a vector with respect to a scalar)**.** Let x be a scalar. The derivative of a vector function $\bar{\mathbf{y}} = \begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix}^{\mathsf{T}}$ with respect to x is the row vector

$$\frac{\partial \bar{\mathbf{y}}}{\partial x} = \begin{bmatrix} \dfrac{\partial y_1}{\partial x} & \dfrac{\partial y_2}{\partial x} & \cdots & \dfrac{\partial y_m}{\partial x} \end{bmatrix}.$$

Definition 10.3 (Derivative of a vector with respect to a vector)**.** The derivative of a vector $\bar{\mathbf{y}} \in \mathbb{R}^m$ with respect to another vector $\bar{\mathbf{x}} \in \mathbb{R}^n$ is the $n \times m$ matrix

$$\frac{\partial \bar{\mathbf{y}}}{\partial \bar{\mathbf{x}}} = \begin{bmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_2}{\partial x_1} & \cdots & \dfrac{\partial y_m}{\partial x_1} \\ \dfrac{\partial y_1}{\partial x_2} & \dfrac{\partial y_2}{\partial x_2} & \cdots & \dfrac{\partial y_m}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial y_1}{\partial x_n} & \dfrac{\partial y_2}{\partial x_n} & \cdots & \dfrac{\partial y_m}{\partial x_n} \end{bmatrix}, \tag{10.2}$$

whose (i, j) entry is $\partial y_j / \partial x_i$.

Remark. These definitions follow the *denominator layout*: rows are indexed by the components of $\bar{\mathbf{x}}$, so that the gradient (10.1) is a column vector. Some texts use the transposed (*numerator*) layout; formulas such as the chain rule then appear with the factors in the opposite order.

Example 10.1 (Derivative of a vector function)

Let

$$\bar{\mathbf{y}} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1^3 + 2x_2 \\ 4x_3 - x_2^2 \end{bmatrix}, \qquad \bar{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Find $\partial \bar{\mathbf{y}} / \partial \bar{\mathbf{x}}$.

Solution.

$$\frac{\partial \bar{\mathbf{y}}}{\partial \bar{\mathbf{x}}} = \begin{bmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_2}{\partial x_1} \\ \dfrac{\partial y_1}{\partial x_2} & \dfrac{\partial y_2}{\partial x_2} \\ \dfrac{\partial y_1}{\partial x_3} & \dfrac{\partial y_2}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 3x_1^2 & 0 \\ 2 & -2x_2 \\ 0 & 4 \end{bmatrix}.$$

◀

10.1.1 Two key results

Theorem 10.4 (Derivative of a linear function)

Let $y = \bar{\mathbf{c}}^{\mathsf{T}}\bar{\mathbf{x}} = c_1x_1 + c_2x_2 + \cdots + c_nx_n = \bar{\mathbf{x}}^{\mathsf{T}}\bar{\mathbf{c}}$. *Then*

$$\frac{\partial y}{\partial \bar{\mathbf{x}}} = \frac{\partial}{\partial \bar{\mathbf{x}}}\bar{\mathbf{c}}^{\mathsf{T}}\bar{\mathbf{x}} = \bar{\mathbf{c}}. \tag{10.3}$$

Proof.

$$\frac{\partial y}{\partial \bar{\mathbf{x}}} = \begin{bmatrix} \frac{\partial y}{\partial x_1} \\ \frac{\partial y}{\partial x_2} \\ \vdots \\ \frac{\partial y}{\partial x_n} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \bar{\mathbf{c}}.$$

■

Theorem 10.5 (Derivative of a quadratic form)

Let

$$y = \bar{\mathbf{x}}^{\mathsf{T}}\mathbf{A}\bar{\mathbf{x}} = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i a_{ij} x_j.$$

Then

$$\frac{\partial y}{\partial \bar{\mathbf{x}}} = \mathbf{A}\bar{\mathbf{x}} + \mathbf{A}^{\mathsf{T}}\bar{\mathbf{x}} = \left(\mathbf{A} + \mathbf{A}^{\mathsf{T}}\right)\bar{\mathbf{x}}. \tag{10.4}$$

When $\mathbf{A} = \mathbf{A}^{\mathsf{T}}$ *is symmetric,*

$$\frac{\partial y}{\partial \bar{\mathbf{x}}} = \mathbf{A}\bar{\mathbf{x}} + \mathbf{A}^{\mathsf{T}}\bar{\mathbf{x}} = 2\mathbf{A}\bar{\mathbf{x}}.$$

Proof. By the product rule, differentiate with respect to each occurrence of $\bar{\mathbf{x}}$ in turn, holding the other fixed. Treating $\mathbf{A}\bar{\mathbf{x}}$ as a constant vector in the first term and $\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{A}$ as a constant row vector in the second, and using Theorem 10.4,

$$\frac{\partial y}{\partial \bar{\mathbf{x}}} = \frac{\partial}{\partial \bar{\mathbf{x}}}\bar{\mathbf{x}}^{\mathsf{T}}(\mathbf{A}\bar{\mathbf{x}}) + \frac{\partial}{\partial \bar{\mathbf{x}}}(\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{A})\bar{\mathbf{x}} = \mathbf{A}\bar{\mathbf{x}} + (\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{A})^{\mathsf{T}} = \mathbf{A}\bar{\mathbf{x}} + \mathbf{A}^{\mathsf{T}}\bar{\mathbf{x}}.$$

■

Example 10.2 (Gradient of a least-squares cost)

Find the gradient of $f(\bar{\mathbf{w}}) = \|\bar{\mathbf{y}} - \mathbf{X}\bar{\mathbf{w}}\|^2$ with respect to $\bar{\mathbf{w}}$, and set it to zero.

Solution. Expanding, $f(\bar{\mathbf{w}}) = \bar{\mathbf{y}}^{\mathsf{T}}\bar{\mathbf{y}} - 2(\mathbf{X}^{\mathsf{T}}\bar{\mathbf{y}})^{\mathsf{T}}\bar{\mathbf{w}} + \bar{\mathbf{w}}^{\mathsf{T}}(\mathbf{X}^{\mathsf{T}}\mathbf{X})\bar{\mathbf{w}}$. By Theorems 10.4 and 10.5, with $\mathbf{X}^{\mathsf{T}}\mathbf{X}$ symmetric,

$$\frac{\partial f}{\partial \bar{\mathbf{w}}} = -2\mathbf{X}^{\mathsf{T}}\bar{\mathbf{y}} + 2\mathbf{X}^{\mathsf{T}}\mathbf{X}\bar{\mathbf{w}}.$$

Setting this to zero gives the normal equations

$$\mathbf{X}^{\mathsf{T}}\mathbf{X}\bar{\mathbf{w}} = \mathbf{X}^{\mathsf{T}}\bar{\mathbf{y}} \;\Rightarrow\; \bar{\mathbf{w}} = (\mathbf{X}^{\mathsf{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathsf{T}}\bar{\mathbf{y}},$$

when $\mathbf{X}$ has full column rank. ◂

10.2 The Jacobian

Definition 10.6 (Jacobian). In multivariate analysis, suppose $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ have the same order n, and consider the transformation $\bar{\mathbf{y}} = \bar{\mathbf{y}}(\bar{\mathbf{x}})$. The **Jacobian** of this transformation is

$$J = \left|\left(\frac{\partial \bar{\mathbf{y}}}{\partial \bar{\mathbf{x}}}\right)^{\mathsf{T}}\right| = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}. \tag{10.5}$$

Since $|\mathbf{M}^{\mathsf{T}}| = |\mathbf{M}|$, the Jacobian also equals $|\partial \bar{\mathbf{y}}/\partial \bar{\mathbf{x}}|$. Geometrically, $|J|$ is the factor by which the transformation scales small volumes (Chapter 5).

Example 10.3 (Spherical coordinates)

The transformation from spherical to Cartesian coordinates (Figure 10.1) is

$$\begin{aligned} x &= r\sin\theta\cos\phi = x(r,\theta,\phi), \\ y &= r\sin\theta\sin\phi = y(r,\theta,\phi), \\ z &= r\cos\theta = z(r,\theta,\phi), \end{aligned}$$

with $r > 0$, $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$. Find its Jacobian.

Solution. The Jacobian is

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}.$$

Expanding along the third row, this simplifies to

$$\begin{aligned} J &= \cos\theta\left(r^2\cos\theta\sin\theta\cos^2\phi + r^2\cos\theta\sin\theta\sin^2\phi\right) \\ &\quad + r\sin\theta\left(r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi\right) \\ &= r^2\cos^2\theta\sin\theta + r^2\sin^3\theta = r^2\sin\theta. \end{aligned}$$

This is why the volume element in spherical coordinates is

$$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi.$$

◂

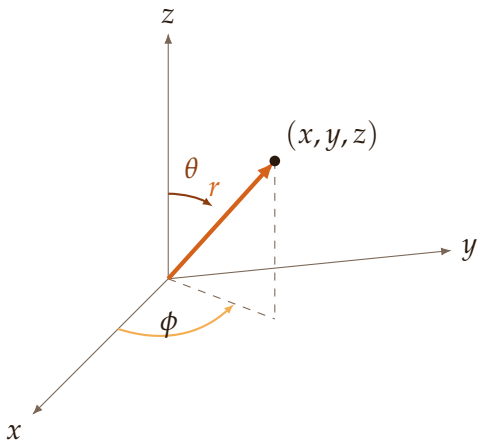


Figure 10.1 Spherical coordinates (r, θ, ϕ) of a point (x, y, z): r is the distance from the origin, θ the angle from the z-axis, and ϕ the angle of the projection on the xy-plane from the x-axis.

10.3 The chain rule

Theorem 10.7 (Chain rule)

Let

$$\bar{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \bar{\mathbf{y}} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}, \qquad \bar{\mathbf{z}} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_r \end{bmatrix},$$

where $\bar{\mathbf{y}}$ is a function of $\bar{\mathbf{x}}$ and $\bar{\mathbf{z}}$ is a function of $\bar{\mathbf{y}}$. Then

$$\frac{\partial \bar{\mathbf{z}}}{\partial \bar{\mathbf{x}}} = \frac{\partial \bar{\mathbf{y}}}{\partial \bar{\mathbf{x}}} \frac{\partial \bar{\mathbf{z}}}{\partial \bar{\mathbf{y}}}. \tag{10.6}$$

If $\bar{\mathbf{w}}$ is in turn a function of $\bar{\mathbf{z}}$, the chain rule extends as

$$\frac{\partial \bar{\mathbf{w}}}{\partial \bar{\mathbf{x}}} = \frac{\partial \bar{\mathbf{y}}}{\partial \bar{\mathbf{x}}} \frac{\partial \bar{\mathbf{z}}}{\partial \bar{\mathbf{y}}} \frac{\partial \bar{\mathbf{w}}}{\partial \bar{\mathbf{z}}}.$$

Proof. By the scalar chain rule,

$$\left[\frac{\partial \bar{\mathbf{z}}}{\partial \bar{\mathbf{x}}}\right]_{ji} = \frac{\partial z_i}{\partial x_j} = \sum_{k=1}^{m} \frac{\partial z_i}{\partial y_k} \frac{\partial y_k}{\partial x_j} = \sum_{k=1}^{m} \left[\frac{\partial \bar{\mathbf{z}}}{\partial \bar{\mathbf{y}}}\right]_{ki} \left[\frac{\partial \bar{\mathbf{y}}}{\partial \bar{\mathbf{x}}}\right]_{jk}.$$

This can be written as

$$\left[\left(\frac{\partial \bar{\mathbf{z}}}{\partial \bar{\mathbf{x}}}\right)^{\mathsf{T}}\right]_{ij} = \left[\frac{\partial \bar{\mathbf{z}}}{\partial \bar{\mathbf{x}}}\right]_{ji} = \sum_{k=1}^{m} \left[\left(\frac{\partial \bar{\mathbf{z}}}{\partial \bar{\mathbf{y}}}\right)^{\mathsf{T}}\right]_{ik} \left[\left(\frac{\partial \bar{\mathbf{y}}}{\partial \bar{\mathbf{x}}}\right)^{\mathsf{T}}\right]_{kj}.$$

Therefore

$$\left(\frac{\partial \bar{\mathbf{z}}}{\partial \bar{\mathbf{x}}}\right)^{\mathsf{T}} = \left(\frac{\partial \bar{\mathbf{z}}}{\partial \bar{\mathbf{y}}}\right)^{\mathsf{T}} \left(\frac{\partial \bar{\mathbf{y}}}{\partial \bar{\mathbf{x}}}\right)^{\mathsf{T}} = \left(\frac{\partial \bar{\mathbf{y}}}{\partial \bar{\mathbf{x}}} \frac{\partial \bar{\mathbf{z}}}{\partial \bar{\mathbf{y}}}\right)^{\mathsf{T}} \Rightarrow \frac{\partial \bar{\mathbf{z}}}{\partial \bar{\mathbf{x}}} = \frac{\partial \bar{\mathbf{y}}}{\partial \bar{\mathbf{x}}} \frac{\partial \bar{\mathbf{z}}}{\partial \bar{\mathbf{y}}}.$$

Applying the result twice gives the extension to $\bar{\mathbf{w}}$. ∎

AIML & 6G Connection

Backpropagation in neural networks is the chain rule of Theorem 10.7 applied layer by layer. For a network $\bar{\mathbf{x}} \to \bar{\mathbf{h}}_1 \to \bar{\mathbf{h}}_2 \to \cdots \to L$ with scalar loss L, the gradient $\frac{\partial L}{\partial \bar{\mathbf{x}}} = \frac{\partial \bar{\mathbf{h}}_1}{\partial \bar{\mathbf{x}}} \frac{\partial \bar{\mathbf{h}}_2}{\partial \bar{\mathbf{h}}_1} \cdots \frac{\partial L}{\partial \bar{\mathbf{h}}_k}$ is a product of matrices, evaluated from the right (the loss) to the left. This is why training is dominated by matrix multiplications.

10.4 Matrix derivatives

Definition 10.8 (Derivative with respect to a matrix). Let $\mathbf{X}$ be a matrix of size $m \times n$ and let $y = f(\mathbf{X})$ be a scalar function of $\mathbf{X}$. The derivative of y with respect to $\mathbf{X}$ is the $m \times n$ matrix

$$\frac{\partial y}{\partial \mathbf{X}} = \begin{bmatrix} \frac{\partial y}{\partial x_{11}} & \frac{\partial y}{\partial x_{12}} & \cdots & \frac{\partial y}{\partial x_{1n}} \\ \frac{\partial y}{\partial x_{21}} & \frac{\partial y}{\partial x_{22}} & \cdots & \frac{\partial y}{\partial x_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y}{\partial x_{m1}} & \frac{\partial y}{\partial x_{m2}} & \cdots & \frac{\partial y}{\partial x_{mn}} \end{bmatrix}. \tag{10.7}$$

10.4.1 Derivative of the trace

Theorem 10.9 (Derivative of the trace)

For a square matrix $\mathbf{X}$,

$$\frac{\partial}{\partial \mathbf{X}} \operatorname{tr}(\mathbf{X}) = \mathbf{I}. \tag{10.8}$$

Proof. The trace is

$$y = \operatorname{tr}(\mathbf{X}) = \sum_{i=1}^{n} x_{ii} = x_{11} + x_{22} + \cdots + x_{nn}.$$

Observe that

$$\frac{\partial}{\partial x_{ii}} \operatorname{tr}(\mathbf{X}) = 1, \qquad \frac{\partial}{\partial x_{ij}} \operatorname{tr}(\mathbf{X}) = 0, \quad i \neq j,$$

so $\frac{\partial}{\partial \mathbf{X}} \operatorname{tr}(\mathbf{X}) = \mathbf{I}$. ∎

10.4.2 Derivative of the determinant

Theorem 10.10 (Derivative of the determinant)

For a square matrix $\mathbf{X}$ *with cofactor matrix* $\mathbf{C} = [C_{ij}]$,

$$\frac{\partial}{\partial \mathbf{X}} |\mathbf{X}| = \mathbf{C} = \left(|\mathbf{X}| \mathbf{X}^{-1} \right)^{\mathsf{T}}, \tag{10.9}$$

where the second form holds when $\mathbf{X}$ *is invertible.*

Proof. By the Laplace expansion along row i (Chapter 5),

$$|\mathbf{X}| = x_{i1}C_{i1} + x_{i2}C_{i2} + \cdots + x_{in}C_{in} = \sum_{j=1}^{n} x_{ij}C_{ij}.$$

None of the cofactors $C_{i1}, \ldots, C_{in}$ involve row i, so

$$\frac{\partial}{\partial x_{ij}} |\mathbf{X}| = C_{ij}.$$

Therefore

$$\frac{\partial}{\partial \mathbf{X}} |\mathbf{X}| = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix} = \mathbf{C},$$

where C_{ij} is the cofactor of x_{ij}. Recall that $\mathbf{X}^{-1} = \dfrac{\operatorname{adj}(\mathbf{X})}{|\mathbf{X}|} = \dfrac{\mathbf{C}^{\mathsf{T}}}{|\mathbf{X}|}$, so $\mathbf{C} = \left(|\mathbf{X}| \mathbf{X}^{-1} \right)^{\mathsf{T}}$. Therefore

$$\frac{\partial}{\partial \mathbf{X}} |\mathbf{X}| = \mathbf{C} = \left(|\mathbf{X}| \mathbf{X}^{-1} \right)^{\mathsf{T}}.$$

■

Example 10.4 (A 2×2 determinant)

Verify Theorem 10.10 for $\mathbf{X} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Solution. Here $|\mathbf{X}| = ad - bc$, so differentiating entry by entry,

$$\frac{\partial}{\partial \mathbf{X}} |\mathbf{X}| = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}.$$

On the other hand, $|\mathbf{X}| \mathbf{X}^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, whose transpose is the same matrix. ◂

Remark. Combining Theorem 10.10 with the scalar chain rule gives $\dfrac{\partial}{\partial \mathbf{X}} \ln |\mathbf{X}| = \dfrac{1}{|\mathbf{X}|} \left(|\mathbf{X}| \mathbf{X}^{-1} \right)^{\mathsf{T}} = \left(\mathbf{X}^{-1} \right)^{\mathsf{T}}$ when $|\mathbf{X}| > 0$. This log-determinant derivative appears in maximum-likelihood estimation of Gaussian covariance matrices and in optimizing the

MIMO capacity $\log_2 \det\left(\mathbf{I} + \mathbf{H}\mathbf{Q}\mathbf{H}^{\mathsf{H}}\right)$.

Function	Derivative
$\bar{\mathbf{c}}^{\mathsf{T}}\bar{\mathbf{x}}$	$\bar{\mathbf{c}}$
$\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{A}\bar{\mathbf{x}}$	$(\mathbf{A}+\mathbf{A}^{\mathsf{T}})\bar{\mathbf{x}}$
$\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{A}\bar{\mathbf{x}}$, $\mathbf{A}$ symmetric	$2\mathbf{A}\bar{\mathbf{x}}$
$\mathrm{tr}(\mathbf{X})$	$\mathbf{I}$
$\lvert\mathbf{X}\rvert$	$\mathbf{C} = (\lvert\mathbf{X}\rvert\mathbf{X}^{-1})^{\mathsf{T}}$
$\ln\lvert\mathbf{X}\rvert$	$(\mathbf{X}^{-1})^{\mathsf{T}}$

Table 10.1 Common vector and matrix derivatives (denominator layout).

Chapter summary

- The gradient $\partial y/\partial\bar{\mathbf{x}}$ collects the partial derivatives; $\partial\bar{\mathbf{y}}/\partial\bar{\mathbf{x}}$ has entries $\partial y_j/\partial x_i$.
- $\frac{\partial}{\partial\bar{\mathbf{x}}}\bar{\mathbf{c}}^{\mathsf{T}}\bar{\mathbf{x}} = \bar{\mathbf{c}}$ and $\frac{\partial}{\partial\bar{\mathbf{x}}}\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{A}\bar{\mathbf{x}} = (\mathbf{A}+\mathbf{A}^{\mathsf{T}})\bar{\mathbf{x}}$, which is $2\mathbf{A}\bar{\mathbf{x}}$ for symmetric $\mathbf{A}$.
- The Jacobian is the determinant of the derivative matrix; for spherical coordinates it is $r^2\sin\theta$.
- Chain rule: $\frac{\partial\bar{\mathbf{z}}}{\partial\bar{\mathbf{x}}} = \frac{\partial\bar{\mathbf{y}}}{\partial\bar{\mathbf{x}}}\frac{\partial\bar{\mathbf{z}}}{\partial\bar{\mathbf{y}}}$.
- $\frac{\partial}{\partial\mathbf{X}}\mathrm{tr}(\mathbf{X}) = \mathbf{I}$ and $\frac{\partial}{\partial\mathbf{X}}\lvert\mathbf{X}\rvert = (\lvert\mathbf{X}\rvert\mathbf{X}^{-1})^{\mathsf{T}}$.

Exercises

10.1 Find $\partial\bar{\mathbf{y}}/\partial\bar{\mathbf{x}}$ for $\bar{\mathbf{y}} = \begin{bmatrix} x_1x_2 & x_1+x_2^2 & \sin x_3 \end{bmatrix}^{\mathsf{T}}$.

10.2 Show that $\frac{\partial}{\partial\bar{\mathbf{x}}}\|\bar{\mathbf{x}}\|^2 = 2\bar{\mathbf{x}}$.

10.3 Compute $\frac{\partial}{\partial\bar{\mathbf{x}}}\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{A}\bar{\mathbf{x}}$ for $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, and check your answer by expanding $\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{A}\bar{\mathbf{x}}$ directly.

10.4 Find the Jacobian of the polar-coordinate transformation $x = r\cos\theta$, $y = r\sin\theta$.

10.5 Show that $\frac{\partial}{\partial\mathbf{X}}\mathrm{tr}(\mathbf{A}\mathbf{X}) = \mathbf{A}^{\mathsf{T}}$.

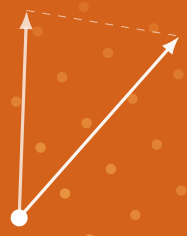
10.6 Show that $\frac{\partial}{\partial\mathbf{X}}\ln\lvert\mathbf{X}\rvert = (\mathbf{X}^{-1})^{\mathsf{T}}$ for $\lvert\mathbf{X}\rvert > 0$.

10.7 Verify Theorem 10.10 numerically for $\mathbf{X} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$.

10.8 Use the chain rule to find $\partial L/\partial\bar{\mathbf{x}}$ for $L = \|\mathbf{W}\bar{\mathbf{x}} - \bar{\mathbf{b}}\|^2$.

11

Least Squares



Overdetermined systems, the normal equations, and the geometry of least squares

In this chapter

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11.2 Geometric intuition 138
11.3 Worked example 139

11.1 The least squares problem

Definition 11.1 (Tall matrix and overdetermined system). Let $\mathbf{A}$ be an $m \times n$ matrix with $m > n$. Such a matrix is called a **tall matrix**. The system of equations

$$\bar{\mathbf{b}} = \mathbf{A}\bar{\mathbf{x}}$$

has more equations than unknowns, since $m > n$, and is called an **overdetermined system**. Such a system typically has no solution.

Since there is usually no exact solution, we instead find the $\bar{\mathbf{x}}$ that minimizes the error

$$\bar{\mathbf{e}} = \bar{\mathbf{b}} - \mathbf{A}\bar{\mathbf{x}}.$$

This can be formulated as the problem

$$\hat{\bar{\mathbf{x}}} = \arg\min_{\bar{\mathbf{x}}} \|\bar{\mathbf{b}} - \mathbf{A}\bar{\mathbf{x}}\|^2, \tag{11.1}$$

which is called the **least squares problem**.

11.1.1 The least squares solution

First, expand the cost function:

$$\begin{aligned}\|\bar{\mathbf{b}} - \mathbf{A}\bar{\mathbf{x}}\|^2 &= (\bar{\mathbf{b}} - \mathbf{A}\bar{\mathbf{x}})^\mathsf{T}(\bar{\mathbf{b}} - \mathbf{A}\bar{\mathbf{x}}) \\ &= (\bar{\mathbf{b}}^\mathsf{T} - \bar{\mathbf{x}}^\mathsf{T}\mathbf{A}^\mathsf{T})(\bar{\mathbf{b}} - \mathbf{A}\bar{\mathbf{x}}) \\ &= \bar{\mathbf{b}}^\mathsf{T}\bar{\mathbf{b}} - \bar{\mathbf{x}}^\mathsf{T}\mathbf{A}^\mathsf{T}\bar{\mathbf{b}} - \bar{\mathbf{b}}^\mathsf{T}\mathbf{A}\bar{\mathbf{x}} + \bar{\mathbf{x}}^\mathsf{T}\mathbf{A}^\mathsf{T}\mathbf{A}\bar{\mathbf{x}}.\end{aligned}$$

To find the least squares solution, set the gradient to zero,

$$\frac{\partial}{\partial \bar{\mathbf{x}}}\|\bar{\mathbf{b}} - \mathbf{A}\bar{\mathbf{x}}\|^2 = \mathbf{0}.$$

Using the derivatives of linear and quadratic forms (Chapter 10), with $\mathbf{A}^\mathsf{T}\mathbf{A}$ symmetric,

$$\begin{aligned}\frac{\partial}{\partial \bar{\mathbf{x}}}\|\bar{\mathbf{b}} - \mathbf{A}\bar{\mathbf{x}}\|^2 &= \frac{\partial}{\partial \bar{\mathbf{x}}}\left(\bar{\mathbf{b}}^\mathsf{T}\bar{\mathbf{b}} - \bar{\mathbf{x}}^\mathsf{T}\mathbf{A}^\mathsf{T}\bar{\mathbf{b}} - \bar{\mathbf{b}}^\mathsf{T}\mathbf{A}\bar{\mathbf{x}} + \bar{\mathbf{x}}^\mathsf{T}\mathbf{A}^\mathsf{T}\mathbf{A}\bar{\mathbf{x}}\right) \\ &= \mathbf{0} - \mathbf{A}^\mathsf{T}\bar{\mathbf{b}} - (\bar{\mathbf{b}}^\mathsf{T}\mathbf{A})^\mathsf{T} + 2\mathbf{A}^\mathsf{T}\mathbf{A}\bar{\mathbf{x}} \\ &= 2\mathbf{A}^\mathsf{T}\mathbf{A}\bar{\mathbf{x}} - 2\mathbf{A}^\mathsf{T}\bar{\mathbf{b}} = \mathbf{0}.\end{aligned}$$

Theorem 11.2 (Normal equations and least squares solution)

The least squares solution $\hat{\mathbf{x}}$ *satisfies the* ***normal equations***

$$2\mathbf{A}^\mathsf{T}\mathbf{A}\hat{\mathbf{x}} - 2\mathbf{A}^\mathsf{T}\bar{\mathbf{b}} = \mathbf{0} \;\Rightarrow\; \mathbf{A}^\mathsf{T}\mathbf{A}\hat{\mathbf{x}} = \mathbf{A}^\mathsf{T}\bar{\mathbf{b}}.$$

If $\mathbf{A}$ *has full column rank,* $\mathbf{A}^\mathsf{T}\mathbf{A}$ *is invertible and*

$$\hat{\mathbf{x}} = \left(\mathbf{A}^\mathsf{T}\mathbf{A}\right)^{-1}\mathbf{A}^\mathsf{T}\bar{\mathbf{b}}. \tag{11.2}$$

Remark. The matrix $(\mathbf{A}^\mathsf{T}\mathbf{A})^{-1}\mathbf{A}^\mathsf{T}$ is the pseudoinverse $\mathbf{A}^+$ of a tall matrix with full column rank (Chapter 9), so $\hat{\mathbf{x}} = \mathbf{A}^+\bar{\mathbf{b}}$. Since $\mathbf{A}^\mathsf{T}\mathbf{A}$ is positive semidefinite, the cost is a convex quadratic, and the stationary point $\hat{\mathbf{x}}$ is its global minimum.

11.2 Geometric intuition

There is also an intuitive way to arrive at the least squares solution. For any $\bar{\mathbf{x}}$, the vector $\mathbf{A}\bar{\mathbf{x}}$ lies in the column space of $\mathbf{A}$, and $\bar{\mathbf{e}} = \bar{\mathbf{b}} - \mathbf{A}\bar{\mathbf{x}}$ is the error in approximating $\bar{\mathbf{b}}$. The error is smallest when it is perpendicular to colsp$(\mathbf{A})$ (Figure 11.1).

Therefore the minimum error must be perpendicular to each of the columns $\bar{\mathbf{a}}_1, \ldots, \bar{\mathbf{a}}_n$ of $\mathbf{A}$:

$$\bar{\mathbf{a}}_1^\mathsf{T}\bar{\mathbf{e}} = 0, \qquad \bar{\mathbf{a}}_2^\mathsf{T}\bar{\mathbf{e}} = 0, \qquad \ldots, \qquad \bar{\mathbf{a}}_n^\mathsf{T}\bar{\mathbf{e}} = 0.$$

These conditions can be combined as

$$\begin{bmatrix}\bar{\mathbf{a}}_1^\mathsf{T} \\ \bar{\mathbf{a}}_2^\mathsf{T} \\ \vdots \\ \bar{\mathbf{a}}_n^\mathsf{T}\end{bmatrix}\bar{\mathbf{e}} = \mathbf{0} \;\Rightarrow\; \mathbf{A}^\mathsf{T}\bar{\mathbf{e}} = \mathbf{0}.$$

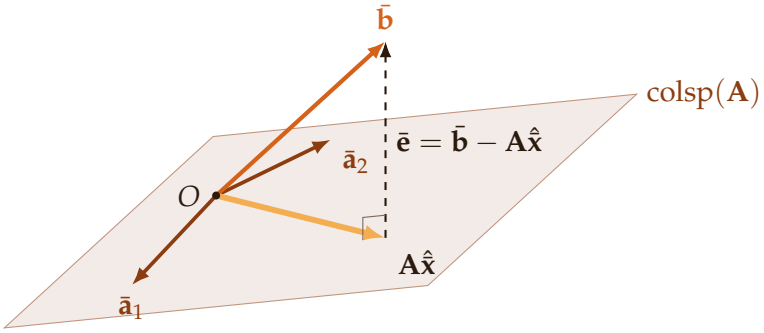


Figure 11.1 Least squares as projection: $\mathbf{A}\hat{\bar{\mathbf{x}}}$ is the point of $\mathrm{colsp}(\mathbf{A})$ closest to $\bar{\mathbf{b}}$, and the error $\bar{\mathbf{e}} = \bar{\mathbf{b}} - \mathbf{A}\hat{\bar{\mathbf{x}}}$ is perpendicular to the column space.

Therefore

$$\begin{aligned} \mathbf{A}^\mathsf{T}(\bar{\mathbf{b}} - \mathbf{A}\hat{\bar{\mathbf{x}}}) = \mathbf{0} \;&\Rightarrow\; \mathbf{A}^\mathsf{T}\bar{\mathbf{b}} - \mathbf{A}^\mathsf{T}\mathbf{A}\hat{\bar{\mathbf{x}}} = \mathbf{0} \\ &\Rightarrow\; \mathbf{A}^\mathsf{T}\mathbf{A}\hat{\bar{\mathbf{x}}} = \mathbf{A}^\mathsf{T}\bar{\mathbf{b}} \\ &\Rightarrow\; \hat{\bar{\mathbf{x}}} = \left(\mathbf{A}^\mathsf{T}\mathbf{A}\right)^{-1}\mathbf{A}^\mathsf{T}\bar{\mathbf{b}}, \end{aligned}$$

the same normal equations and solution as in Theorem 11.2. In the language of Chapter 6, the error lies in the null space of $\mathbf{A}^\mathsf{T}$, the orthogonal complement of the column space.

> **Remark.** The vector $\mathbf{A}\hat{\bar{\mathbf{x}}} = \mathbf{A}(\mathbf{A}^\mathsf{T}\mathbf{A})^{-1}\mathbf{A}^\mathsf{T}\bar{\mathbf{b}}$ is the orthogonal projection of $\bar{\mathbf{b}}$ onto $\mathrm{colsp}(\mathbf{A})$, and $\mathbf{P} = \mathbf{A}(\mathbf{A}^\mathsf{T}\mathbf{A})^{-1}\mathbf{A}^\mathsf{T}$ is the **projection matrix**. For a single column $\mathbf{A} = \bar{\mathbf{v}}$, this reduces to the projection $\frac{\bar{\mathbf{v}}^\mathsf{T}\bar{\mathbf{b}}}{\bar{\mathbf{v}}^\mathsf{T}\bar{\mathbf{v}}}\bar{\mathbf{v}}$ of Chapter 2.

11.3 Worked example

Example 11.1 (A 4×2 least squares problem)

Solve the least squares problem below and calculate $\hat{\bar{\mathbf{x}}}$:

$$\min_{\bar{\mathbf{x}}} \| \underbrace{\begin{bmatrix} 2 \\ 0 \\ 3 \\ -1 \end{bmatrix}}_{\bar{\mathbf{b}}} - \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}}_{\mathbf{A}} \bar{\mathbf{x}} \|^2.$$

Solution. We start with

$$\mathbf{A}^\mathsf{T}\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix}.$$

Next,

$$\left(\mathbf{A}^\mathsf{T}\mathbf{A}\right)^{-1} = \begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix}^{-1} = \frac{1}{20}\begin{bmatrix} 14 & -6 \\ -6 & 4 \end{bmatrix} = \begin{bmatrix} \frac{7}{10} & -\frac{3}{10} \\ -\frac{3}{10} & \frac{1}{5} \end{bmatrix}.$$

Subsequently,

$$\left(\mathbf{A}^{\mathsf{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathsf{T}} = \frac{1}{20}\begin{bmatrix}14 & -6\\ -6 & 4\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 2 & 3\end{bmatrix} = \frac{1}{20}\begin{bmatrix}14 & 8 & 2 & -4\\ -6 & -2 & 2 & 6\end{bmatrix}.$$

Finally,

$$\hat{\bar{\mathbf{x}}} = \left(\mathbf{A}^{\mathsf{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathsf{T}}\bar{\mathbf{b}} = \frac{1}{20}\begin{bmatrix}14 & 8 & 2 & -4\\ -6 & -2 & 2 & 6\end{bmatrix}\begin{bmatrix}2\\ 0\\ 3\\ -1\end{bmatrix} = \frac{1}{20}\begin{bmatrix}38\\ -12\end{bmatrix} = \begin{bmatrix}\frac{19}{10}\\ -\frac{3}{5}\end{bmatrix}.$$

Check. The fitted vector and error are

$$\mathbf{A}\hat{\bar{\mathbf{x}}} = \begin{bmatrix}1.9\\ 1.3\\ 0.7\\ 0.1\end{bmatrix}, \qquad \bar{\mathbf{e}} = \bar{\mathbf{b}} - \mathbf{A}\hat{\bar{\mathbf{x}}} = \begin{bmatrix}0.1\\ -1.3\\ 2.3\\ -1.1\end{bmatrix},$$

and $\mathbf{A}^{\mathsf{T}}\bar{\mathbf{e}} = \begin{bmatrix}0.1-1.3+2.3-1.1\\ -1.3+4.6-3.3\end{bmatrix} = \mathbf{0}$: the error is orthogonal to both columns of $\mathbf{A}$, as the geometry predicts. ◂

Remark. This example is a straight-line fit: with $\bar{\mathbf{x}} = \begin{bmatrix}c & d\end{bmatrix}^{\mathsf{T}}$, the rows of $\mathbf{A}\bar{\mathbf{x}} \approx \bar{\mathbf{b}}$ read $c + dt_i \approx b_i$ at $t_i = 0, 1, 2, 3$. The best line is $b = 1.9 - 0.6t$.

AIML & 6G Connection

Least squares is linear regression: with data matrix $\mathbf{X}$ and targets $\bar{\mathbf{y}}$, the weights are $\hat{\bar{\mathbf{w}}} = (\mathbf{X}^{\mathsf{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathsf{T}}\bar{\mathbf{y}}$. In wireless receivers, a known pilot matrix $\mathbf{X}$ is sent, and the channel is estimated from $\bar{\mathbf{y}} = \mathbf{X}\bar{\mathbf{h}} + \bar{\mathbf{n}}$ as $\hat{\bar{\mathbf{h}}} = (\mathbf{X}^{\mathsf{H}}\mathbf{X})^{-1}\mathbf{X}^{\mathsf{H}}\bar{\mathbf{y}}$. This is the complex least squares solution, with $^{\mathsf{T}}$ replaced by $^{\mathsf{H}}$.

Chapter summary

- A tall $\mathbf{A}$ ($m > n$) gives an overdetermined system $\bar{\mathbf{b}} = \mathbf{A}\bar{\mathbf{x}}$, usually with no exact solution.
- Least squares minimizes the error $\|\bar{\mathbf{b}} - \mathbf{A}\bar{\mathbf{x}}\|^2$.
- Setting the gradient to zero gives the normal equations $\mathbf{A}^{\mathsf{T}}\mathbf{A}\hat{\bar{\mathbf{x}}} = \mathbf{A}^{\mathsf{T}}\bar{\mathbf{b}}$.
- The solution is $\hat{\bar{\mathbf{x}}} = (\mathbf{A}^{\mathsf{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathsf{T}}\bar{\mathbf{b}}$.
- Geometrically, the error $\bar{\mathbf{b}} - \mathbf{A}\hat{\bar{\mathbf{x}}}$ is orthogonal to the column space of $\mathbf{A}$, and $\mathbf{A}\hat{\bar{\mathbf{x}}}$ is the projection of $\bar{\mathbf{b}}$ onto it.

Exercises

11.1 Fit the line $b = c + dt$ to the points $(t, b) = (0, 6), (1, 0), (2, 0)$ by least squares, and verify that the error is orthogonal to the columns of $\mathbf{A}$.

11.2 Compute the minimum squared error $\|\bar{\mathbf{b}} - \mathbf{A}\hat{\bar{\mathbf{x}}}\|^2$ in Example 11.1.

11.3 Show that the projection matrix $\mathbf{P} = \mathbf{A}(\mathbf{A}^\mathsf{T}\mathbf{A})^{-1}\mathbf{A}^\mathsf{T}$ satisfies $\mathbf{P}^2 = \mathbf{P}$ and $\mathbf{P}^\mathsf{T} = \mathbf{P}$, and that $\mathbf{I} - \mathbf{P}$ projects onto $\mathcal{N}(\mathbf{A}^\mathsf{T})$.

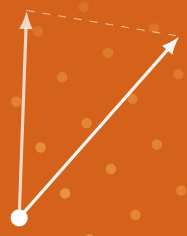
11.4 Show that minimizing $(\bar{\mathbf{b}} - \mathbf{A}\bar{\mathbf{x}})^\mathsf{T}\mathbf{W}(\bar{\mathbf{b}} - \mathbf{A}\bar{\mathbf{x}})$ for a positive definite $\mathbf{W}$ gives $\hat{\bar{\mathbf{x}}} = (\mathbf{A}^\mathsf{T}\mathbf{W}\mathbf{A})^{-1}\mathbf{A}^\mathsf{T}\mathbf{W}\bar{\mathbf{b}}$ (weighted least squares).

11.5 Show that if $\mathbf{A}$ has full column rank, then $\mathbf{A}^\mathsf{T}\mathbf{A}$ is positive definite and hence invertible.

11.6 For complex $\mathbf{A}$ and $\bar{\mathbf{b}}$, show that the minimizer of $\|\bar{\mathbf{b}} - \mathbf{A}\bar{\mathbf{x}}\|^2$ is $\hat{\bar{\mathbf{x}}} = (\mathbf{A}^\mathsf{H}\mathbf{A})^{-1}\mathbf{A}^\mathsf{H}\bar{\mathbf{b}}$.

12

Least Norm Solution



Underdetermined systems, Lagrange multipliers, and the minimum-norm solution

In this chapter

12.1 The least norm problem 143
12.2 Lagrange multiplier solution 144
12.3 Geometric intuition 144
12.4 Worked example 145

12.1 The least norm problem

Definition 12.1 (Wide matrix and underdetermined system)**.** Let $\mathbf{A}$ be an $m \times n$ matrix with $n > m$. Such a matrix is called a **wide matrix**. Throughout this chapter we assume it has full row rank,

$$\operatorname{rank}(\mathbf{A}) = m.$$

The system $\bar{\mathbf{b}} = \mathbf{A}\bar{\mathbf{x}}$ has fewer equations than unknowns and is called an **underdetermined system**. Its nullity is

$$n - m = \text{number of free variables},$$

so it has infinitely many solutions.

Among these infinitely many solutions, we seek the one with minimum norm. This is formulated as the problem

$$\min_{\bar{\mathbf{x}}} \|\bar{\mathbf{x}}\| \quad \text{subject to} \quad \bar{\mathbf{b}} = \mathbf{A}\bar{\mathbf{x}}. \tag{12.1}$$

This is a **constrained optimization problem**. There are m constraints, one corresponding to each equation of the system $\bar{\mathbf{b}} = \mathbf{A}\bar{\mathbf{x}}$.

12.2 Lagrange multiplier solution

Since there are m constraints, we use m **Lagrange multipliers**

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{bmatrix}.$$

Minimizing $\|\bar{\mathbf{x}}\|$ is equivalent to minimizing $\|\bar{\mathbf{x}}\|^2$, which is easier to differentiate. The **Lagrangian** is

$$f = \|\bar{\mathbf{x}}\|^2 + \boldsymbol{\lambda}^{\mathsf{T}}(\bar{\mathbf{b}} - \mathbf{A}\bar{\mathbf{x}}).$$

Its gradient with respect to $\bar{\mathbf{x}}$ (Chapter 10) is

$$\begin{aligned} \frac{\partial f}{\partial \bar{\mathbf{x}}} &= \frac{\partial}{\partial \bar{\mathbf{x}}} \bar{\mathbf{x}}^{\mathsf{T}} \bar{\mathbf{x}} + \frac{\partial}{\partial \bar{\mathbf{x}}} \left(\boldsymbol{\lambda}^{\mathsf{T}} \bar{\mathbf{b}} - \boldsymbol{\lambda}^{\mathsf{T}} \mathbf{A} \bar{\mathbf{x}} \right) \\ &= 2\bar{\mathbf{x}} - (\boldsymbol{\lambda}^{\mathsf{T}} \mathbf{A})^{\mathsf{T}} = 2\bar{\mathbf{x}} - \mathbf{A}^{\mathsf{T}} \boldsymbol{\lambda} = \mathbf{0} \;\Rightarrow\; \bar{\mathbf{x}} = \tfrac{1}{2} \mathbf{A}^{\mathsf{T}} \boldsymbol{\lambda}. \end{aligned}$$

We now use the constraint to determine $\bar{\mathbf{x}}$:

$$\begin{aligned} \bar{\mathbf{b}} = \mathbf{A}\bar{\mathbf{x}} = \mathbf{A} \tfrac{1}{2} \mathbf{A}^{\mathsf{T}} \boldsymbol{\lambda} \;&\Rightarrow\; \boldsymbol{\lambda} = 2 \left(\mathbf{A}\mathbf{A}^{\mathsf{T}} \right)^{-1} \bar{\mathbf{b}} \\ &\Rightarrow\; \bar{\mathbf{x}} = \tfrac{1}{2} \mathbf{A}^{\mathsf{T}} \boldsymbol{\lambda} = \tfrac{1}{2} \mathbf{A}^{\mathsf{T}} 2 \left(\mathbf{A}\mathbf{A}^{\mathsf{T}} \right)^{-1} \bar{\mathbf{b}} = \mathbf{A}^{\mathsf{T}} \left(\mathbf{A}\mathbf{A}^{\mathsf{T}} \right)^{-1} \bar{\mathbf{b}}. \end{aligned}$$

Here $\mathbf{A}\mathbf{A}^{\mathsf{T}}$ is invertible because $\mathbf{A}$ has full row rank.

Theorem 12.2 (Least norm solution)

The least norm solution of the underdetermined system $\bar{\mathbf{b}} = \mathbf{A}\bar{\mathbf{x}}$, *with* $\operatorname{rank}(\mathbf{A}) = m$, *is*

$$\hat{\bar{\mathbf{x}}} = \mathbf{A}^{\mathsf{T}} \left(\mathbf{A}\mathbf{A}^{\mathsf{T}} \right)^{-1} \bar{\mathbf{b}}. \tag{12.2}$$

Remark. The matrix $\mathbf{A}^{\mathsf{T}}(\mathbf{A}\mathbf{A}^{\mathsf{T}})^{-1}$ is the pseudoinverse $\mathbf{A}^{\dagger}$ of a wide matrix with full row rank (Chapter 9), so $\hat{\bar{\mathbf{x}}} = \mathbf{A}^{\dagger}\bar{\mathbf{b}}$. Compare it with the least squares solution $(\mathbf{A}^{\mathsf{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathsf{T}}\bar{\mathbf{b}}$ for tall matrices (Chapter 11): both are $\mathbf{A}^{\dagger}\bar{\mathbf{b}}$.

12.3 Geometric intuition

The least norm solution can also be understood intuitively. Split any $\bar{\mathbf{x}}$ as

$$\bar{\mathbf{x}} = \bar{\mathbf{x}}_{\parallel} + \bar{\mathbf{x}}_{\perp}.$$

- The vector $\bar{\mathbf{x}}_{\parallel}$ is in the column space of $\mathbf{A}^{\mathsf{T}}$ (the row space of $\mathbf{A}$). Therefore $\bar{\mathbf{x}}_{\parallel} = \mathbf{A}^{\mathsf{T}} \boldsymbol{\lambda}$ for some $\boldsymbol{\lambda}$.
- The vector $\bar{\mathbf{x}}_{\perp}$ lies in the space perpendicular to the column space of $\mathbf{A}^{\mathsf{T}}$. Therefore it lies in the null space of $\mathbf{A}$ (Chapter 6), and hence $\mathbf{A}\bar{\mathbf{x}}_{\perp} = \mathbf{0}$.

Consider now the constraint:

$$\begin{aligned}\bar{\mathbf{b}} = \mathbf{A}\bar{\mathbf{x}} = \mathbf{A}(\bar{\mathbf{x}}_{\parallel} + \bar{\mathbf{x}}_{\perp}) = \mathbf{A}\bar{\mathbf{x}}_{\parallel} + \mathbf{A}\bar{\mathbf{x}}_{\perp} = \mathbf{A}\bar{\mathbf{x}}_{\parallel} = \mathbf{A}\mathbf{A}^{\mathsf{T}}\bar{\boldsymbol{\nu}} \\ \Rightarrow \bar{\boldsymbol{\nu}} = \left(\mathbf{A}\mathbf{A}^{\mathsf{T}}\right)^{-1}\bar{\mathbf{b}} \\ \Rightarrow \bar{\mathbf{x}}_{\parallel} = \mathbf{A}^{\mathsf{T}}\left(\mathbf{A}\mathbf{A}^{\mathsf{T}}\right)^{-1}\bar{\mathbf{b}}.\end{aligned}$$

So every solution has the same component $\bar{\mathbf{x}}_{\parallel}$; only $\bar{\mathbf{x}}_{\perp}$ varies. Since $\bar{\mathbf{x}}_{\parallel} \perp \bar{\mathbf{x}}_{\perp}$, the Pythagorean theorem gives

$$\|\bar{\mathbf{x}}\|^2 = \|\bar{\mathbf{x}}_{\parallel}\|^2 + \|\bar{\mathbf{x}}_{\perp}\|^2.$$

Because $\mathbf{A}\bar{\mathbf{x}}_{\perp} = \mathbf{0}$, any value of $\bar{\mathbf{x}}_{\perp}$ satisfies the constraint. Therefore, to minimize $\|\bar{\mathbf{x}}\|$, we set $\bar{\mathbf{x}}_{\perp} = \mathbf{0}$. The least norm solution follows as

$$\hat{\bar{\mathbf{x}}} = \bar{\mathbf{x}}_{\parallel} = \mathbf{A}^{\mathsf{T}}\left(\mathbf{A}\mathbf{A}^{\mathsf{T}}\right)^{-1}\bar{\mathbf{b}},$$

in agreement with Theorem 12.2 (Figure 12.1).

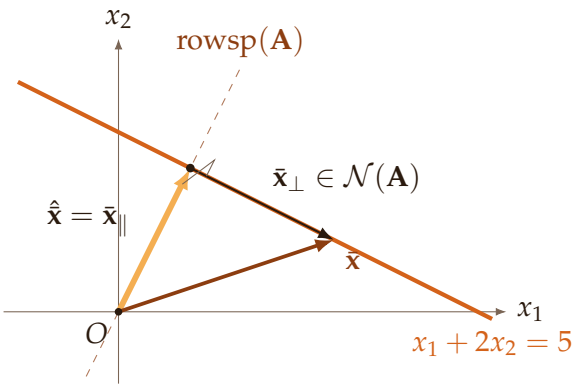


Figure 12.1 For $\mathbf{A} = \begin{bmatrix}1 & 2\end{bmatrix}$ and $b = 5$, the solutions form the line $x_1 + 2x_2 = 5$. Every solution is $\bar{\mathbf{x}} = \bar{\mathbf{x}}_{\parallel} + \bar{\mathbf{x}}_{\perp}$ with the same $\bar{\mathbf{x}}_{\parallel}$ in the row space; the least norm solution $\hat{\bar{\mathbf{x}}} = (1,2)$ has $\bar{\mathbf{x}}_{\perp} = \mathbf{0}$ and is the point of the line closest to the origin.

12.4 Worked example

Example 12.1 (A 2×4 least norm problem)

Solve

$$\min_{\bar{\mathbf{x}}} \|\bar{\mathbf{x}}\| \quad \text{subject to} \quad \begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & 1\\0 & 1 & 2 & 3\end{bmatrix}\bar{\mathbf{x}}.$$

Solution. We start with

$$\mathbf{A}\mathbf{A}^{\mathsf{T}} = \begin{bmatrix}1 & 1 & 1 & 1\\0 & 1 & 2 & 3\end{bmatrix}\begin{bmatrix}1 & 0\\1 & 1\\1 & 2\\1 & 3\end{bmatrix} = \begin{bmatrix}4 & 6\\6 & 14\end{bmatrix}.$$

Next,

$$\left(\mathbf{A}\mathbf{A}^{\mathsf{T}}\right)^{-1} = \begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix}^{-1} = \frac{1}{20}\begin{bmatrix} 14 & -6 \\ -6 & 4 \end{bmatrix} = \begin{bmatrix} \frac{7}{10} & -\frac{3}{10} \\ -\frac{3}{10} & \frac{1}{5} \end{bmatrix}.$$

The solution is

$$\hat{\mathbf{x}} = \mathbf{A}^{\mathsf{T}}\left(\mathbf{A}\mathbf{A}^{\mathsf{T}}\right)^{-1}\bar{\mathbf{b}} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \frac{1}{20}\begin{bmatrix} 14 & -6 \\ -6 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

which evaluates to

$$\hat{\mathbf{x}} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \frac{1}{20}\begin{bmatrix} 22 \\ -8 \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 22 \\ 14 \\ 6 \\ -2 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 11 \\ 7 \\ 3 \\ -1 \end{bmatrix}.$$

Check. $\mathbf{A}\hat{\mathbf{x}} = \frac{1}{10}\begin{bmatrix} 11+7+3-1 \\ 7+6-3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, so the constraint holds, and $\|\hat{\mathbf{x}}\|^2 = \frac{121+49+9+1}{100} = 1.8$. ◂

Remark. The matrix here is the transpose of the matrix in the least squares example of Chapter 11, so the same 2×2 matrix $\begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix}$ appears, once as $\mathbf{A}^{\mathsf{T}}\mathbf{A}$ and once as $\mathbf{A}\mathbf{A}^{\mathsf{T}}$.

	Least squares	Least norm
Shape of $\mathbf{A}$	tall, $m > n$	wide, $n > m$
Rank	n (full column)	m (full row)
Solutions of $\mathbf{A}\bar{\mathbf{x}} = \bar{\mathbf{b}}$	usually none	infinitely many
Objective	$\min \|\bar{\mathbf{b}} - \mathbf{A}\bar{\mathbf{x}}\|^2$	$\min \|\bar{\mathbf{x}}\|$ s.t. $\mathbf{A}\bar{\mathbf{x}} = \bar{\mathbf{b}}$
Solution	$(\mathbf{A}^{\mathsf{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathsf{T}}\bar{\mathbf{b}}$	$\mathbf{A}^{\mathsf{T}}(\mathbf{A}\mathbf{A}^{\mathsf{T}})^{-1}\bar{\mathbf{b}}$
Geometry	error $\perp$ colsp$(\mathbf{A})$	solution $\in$ rowsp$(\mathbf{A})$

Table 12.1 Least squares and least norm compared. Both solutions equal $\mathbf{A}^{\dagger}\bar{\mathbf{b}}$.

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In a downlink with more base-station antennas N_t than single-antenna users K, the transmitter must find a signal $\bar{\mathbf{x}}$ with $\mathbf{H}\bar{\mathbf{x}} = \bar{\mathbf{s}}$, where $\mathbf{H}$ is the wide $K \times N_t$ channel. The least norm choice $\bar{\mathbf{x}} = \mathbf{H}^{\mathsf{H}}(\mathbf{H}\mathbf{H}^{\mathsf{H}})^{-1}\bar{\mathbf{s}}$ delivers each user's symbol with no interference and with the least transmit power: this is *zero-forcing precoding*. In machine learning, an overparameterized linear model has more weights than training samples. Gradient descent started from zero converges to the least norm solution among all the weights that fit the data exactly.

Chapter summary

- A wide $\mathbf{A}$ ($n > m$) with rank m gives an underdetermined system with infinitely many solutions; the nullity $n - m$ is the number of free variables.
- Least norm: minimize $\|\bar{\mathbf{x}}\|$ subject to $\bar{\mathbf{b}} = \mathbf{A}\bar{\mathbf{x}}$, a problem with m constraints.
- Lagrange multipliers give $\bar{\mathbf{x}} = \frac{1}{2}\mathbf{A}^{\mathsf{T}}\bar{\boldsymbol{\nu}}$.
- The solution is $\hat{\bar{\mathbf{x}}} = \mathbf{A}^{\mathsf{T}}(\mathbf{A}\mathbf{A}^{\mathsf{T}})^{-1}\bar{\mathbf{b}}$.
- The null-space part of $\bar{\mathbf{x}}$ does not change the constraint, so it is set to zero; the least norm solution lies in the row space of $\mathbf{A}$.

Exercises

12.1 Find the least norm solution of $x_1 + 2x_2 + 2x_3 = 9$, and its norm.

12.2 Find the least norm solution of $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \bar{\mathbf{x}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and verify that it satisfies the constraints.

12.3 In Example 12.1, find a vector $\bar{\mathbf{n}} \neq \mathbf{0}$ in the null space of $\mathbf{A}$, and verify that $\hat{\bar{\mathbf{x}}} + \bar{\mathbf{n}}$ is a solution with larger norm.

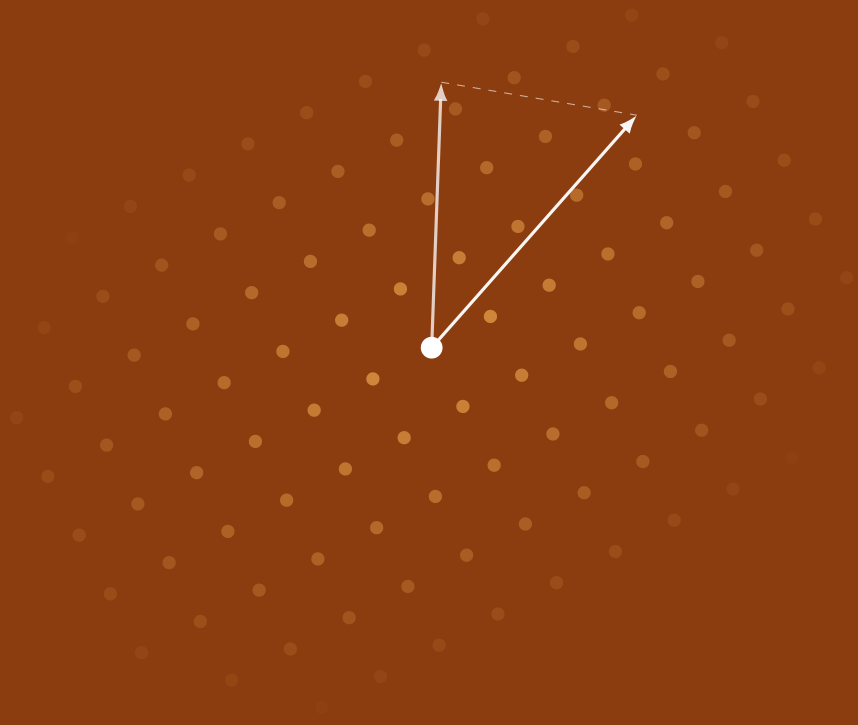
12.4 Show directly that $\hat{\bar{\mathbf{x}}} = \mathbf{A}^{\mathsf{T}}(\mathbf{A}\mathbf{A}^{\mathsf{T}})^{-1}\bar{\mathbf{b}}$ is orthogonal to every vector in the null space of $\mathbf{A}$.

12.5 Show that minimizing $\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{W}\bar{\mathbf{x}}$ subject to $\mathbf{A}\bar{\mathbf{x}} = \bar{\mathbf{b}}$, for positive definite $\mathbf{W}$, gives $\hat{\bar{\mathbf{x}}} = \mathbf{W}^{-1}\mathbf{A}^{\mathsf{T}}(\mathbf{A}\mathbf{W}^{-1}\mathbf{A}^{\mathsf{T}})^{-1}\bar{\mathbf{b}}$.

12.6 Verify that $\mathbf{A}^{\mathsf{T}}(\mathbf{A}\mathbf{A}^{\mathsf{T}})^{-1} = \mathbf{V}\blacksquare^{\dagger}\mathbf{U}^{\mathsf{T}}$ using the SVD of a wide matrix with full row rank.

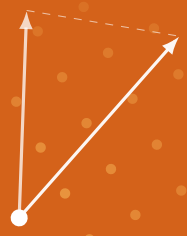
PART VII

WMI, Probability, CH and DTMC



13

Woodbury Matrix Identity



The Woodbury matrix identity, its proof, and inverting low-rank updates

In this chapter

13.1 The Woodbury matrix identity 151
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13.1 The Woodbury matrix identity

Theorem 13.1 (Woodbury matrix identity)

For matrices of compatible sizes, with $\mathbf{A}$, $\mathbf{C}$ *and* $\mathbf{C}^{-1}+\mathbf{V}\mathbf{A}^{-1}\mathbf{U}$ *invertible, the* ***Woodbury matrix identity*** *(WMI) states that*

$$(\mathbf{A}+\mathbf{U}\mathbf{C}\mathbf{V})^{-1}=\mathbf{A}^{-1}-\mathbf{A}^{-1}\mathbf{U}\left(\mathbf{C}^{-1}+\mathbf{V}\mathbf{A}^{-1}\mathbf{U}\right)^{-1}\mathbf{V}\mathbf{A}^{-1}. \tag{13.1}$$

When $\mathbf{A}$ is $n\times n$ and $\mathbf{UCV}$ is a low-rank update (with $\mathbf{C}$ of size $k\times k$, $k\ll n$), the right-hand side only requires inverting the small $k\times k$ matrix $\mathbf{C}^{-1}+\mathbf{V}\mathbf{A}^{-1}\mathbf{U}$, once $\mathbf{A}^{-1}$ is known. The proof rests on two helper identities.

Theorem 13.2 (Identity 1)

If $\mathbf{I}+\mathbf{P}$ *is invertible, then*

$$(\mathbf{I}+\mathbf{P})^{-1}=\mathbf{I}-(\mathbf{I}+\mathbf{P})^{-1}\mathbf{P}. \tag{13.2}$$

Proof.

$$(\mathbf{I}+\mathbf{P})^{-1}=(\mathbf{I}+\mathbf{P})^{-1}\mathbf{I}=(\mathbf{I}+\mathbf{P})^{-1}(\mathbf{I}+\mathbf{P}-\mathbf{P})=\mathbf{I}-(\mathbf{I}+\mathbf{P})^{-1}\mathbf{P}.$$

■

Theorem 13.3 (Identity 2)

If $\mathbf{I}+\mathbf{PQ}$ *and* $\mathbf{I}+\mathbf{QP}$ *are invertible, then*

$$(\mathbf{I}+\mathbf{PQ})^{-1}\,\mathbf{P}=\mathbf{P}\,(\mathbf{I}+\mathbf{QP})^{-1}. \tag{13.3}$$

Proof. Start from the trivial equality $\mathbf{P}+\mathbf{PQP}=\mathbf{P}+\mathbf{PQP}$ and factor each side:

$$\mathbf{P}(\mathbf{I}+\mathbf{QP})=(\mathbf{I}+\mathbf{PQ})\mathbf{P}.$$

Multiplying on the left by $(\mathbf{I}+\mathbf{PQ})^{-1}$ and on the right by $(\mathbf{I}+\mathbf{QP})^{-1}$,

$$(\mathbf{I}+\mathbf{PQ})^{-1}\mathbf{P}=\mathbf{P}(\mathbf{I}+\mathbf{QP})^{-1}.$$

■

Note that $\mathbf{P}$ and $\mathbf{Q}$ need not be square: if $\mathbf{P}$ is $n\times k$ and $\mathbf{Q}$ is $k\times n$, the identity matrices on the two sides have sizes n and k.

13.2 Proof of the Woodbury identity

Proof of Theorem 13.1. First, factor out $\mathbf{A}$:

$$(\mathbf{A}+\mathbf{UCV})^{-1}=\Big(\mathbf{A}(\mathbf{I}+\mathbf{A}^{-1}\mathbf{UCV})\Big)^{-1}=(\mathbf{I}+\mathbf{A}^{-1}\mathbf{UCV})^{-1}\mathbf{A}^{-1}.$$

Use Identity 1 with $\mathbf{P}=\mathbf{A}^{-1}\mathbf{UCV}$:

$$\begin{aligned}(\mathbf{A}+\mathbf{UCV})^{-1}&=(\mathbf{I}+\mathbf{A}^{-1}\mathbf{UCV})^{-1}\mathbf{A}^{-1}\\&=\Big(\mathbf{I}-(\mathbf{I}+\mathbf{A}^{-1}\mathbf{UCV})^{-1}\mathbf{A}^{-1}\mathbf{UCV}\Big)\,\mathbf{A}^{-1}\\&=\mathbf{A}^{-1}-(\mathbf{I}+\mathbf{A}^{-1}\mathbf{UCV})^{-1}\mathbf{A}^{-1}\mathbf{UCVA}^{-1}.\end{aligned}$$

Now use Identity 2 with $\mathbf{P}=\mathbf{A}^{-1}\mathbf{U}$ and $\mathbf{Q}=\mathbf{CV}$, which moves $\mathbf{A}^{-1}\mathbf{U}$ to the left of the inverse:

$$\begin{aligned}(\mathbf{A}+\mathbf{UCV})^{-1}&=\mathbf{A}^{-1}-\mathbf{A}^{-1}\mathbf{U}(\mathbf{I}+\mathbf{CVA}^{-1}\mathbf{U})^{-1}\mathbf{CVA}^{-1}\\&=\mathbf{A}^{-1}-\mathbf{A}^{-1}\mathbf{U}\Big(\mathbf{C}(\mathbf{C}^{-1}+\mathbf{VA}^{-1}\mathbf{U})\Big)^{-1}\mathbf{CVA}^{-1}\\&=\mathbf{A}^{-1}-\mathbf{A}^{-1}\mathbf{U}(\mathbf{C}^{-1}+\mathbf{VA}^{-1}\mathbf{U})^{-1}\mathbf{VA}^{-1},\end{aligned}$$

where the last step uses $(\mathbf{CM})^{-1}\,\mathbf{C}=\mathbf{M}^{-1}\mathbf{C}^{-1}\mathbf{C}=\mathbf{M}^{-1}$. ■

13.3 Application and example

Theorem 13.4 (Rank-one update of the identity)

For any vector $\bar{\mathbf{x}}$,

$$\left(\mathbf{I}+\bar{\mathbf{x}}\bar{\mathbf{x}}^{\mathsf{T}}\right)^{-1}=\mathbf{I}-\frac{\bar{\mathbf{x}}\bar{\mathbf{x}}^{\mathsf{T}}}{1+\|\bar{\mathbf{x}}\|^2}. \tag{13.4}$$

Proof. Apply the WMI with $\mathbf{A}=\mathbf{I}$, $\mathbf{U}=\bar{\mathbf{x}}$, $\mathbf{V}=\bar{\mathbf{x}}^{\mathsf{T}}$ and $\mathbf{C}=1$:

$$\begin{aligned}\left(\mathbf{I}+\bar{\mathbf{x}}\bar{\mathbf{x}}^{\mathsf{T}}\right)^{-1}&=\mathbf{I}-\mathbf{I}\bar{\mathbf{x}}\left(1^{-1}+\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{I}^{-1}\bar{\mathbf{x}}\right)^{-1}\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{I}\\&=\mathbf{I}-\frac{\bar{\mathbf{x}}\bar{\mathbf{x}}^{\mathsf{T}}}{1+\bar{\mathbf{x}}^{\mathsf{T}}\bar{\mathbf{x}}}=\mathbf{I}-\frac{\bar{\mathbf{x}}\bar{\mathbf{x}}^{\mathsf{T}}}{1+\|\bar{\mathbf{x}}\|^2}.\end{aligned}$$

The only inverse needed is of the scalar $1+\|\bar{\mathbf{x}}\|^2$. ■

Example 13.1 (Inverting a rank-one update)

Find the inverse of

$$\mathbf{M}=\begin{bmatrix}2&-1&1&2\\-1&2&-1&-2\\1&-1&2&2\\2&-2&2&5\end{bmatrix}.$$

Solution. The matrix can be expressed as

$$\mathbf{M}=\mathbf{I}+\begin{bmatrix}1\\-1\\1\\2\end{bmatrix}\begin{bmatrix}1&-1&1&2\end{bmatrix}=\mathbf{I}+\bar{\mathbf{x}}\bar{\mathbf{x}}^{\mathsf{T}}.$$

Therefore, using Theorem 13.4 with $\|\bar{\mathbf{x}}\|^2=1+1+1+4=7$,

$$\mathbf{M}^{-1}=\mathbf{I}-\frac{1}{8}\begin{bmatrix}1\\-1\\1\\2\end{bmatrix}\begin{bmatrix}1&-1&1&2\end{bmatrix}=\frac{1}{8}\begin{bmatrix}7&1&-1&-2\\1&7&1&2\\-1&1&7&-2\\-2&2&-2&4\end{bmatrix}.$$

A 4×4 inverse has been found with a single scalar division. ◂

Remark. With $\mathbf{C}=1$, $\mathbf{U}=\bar{\mathbf{u}}$ and $\mathbf{V}=\bar{\mathbf{v}}^{\mathsf{T}}$, the WMI becomes the **Sherman–Morrison formula** $\left(\mathbf{A}+\bar{\mathbf{u}}\bar{\mathbf{v}}^{\mathsf{T}}\right)^{-1}=\mathbf{A}^{-1}-\dfrac{\mathbf{A}^{-1}\bar{\mathbf{u}}\bar{\mathbf{v}}^{\mathsf{T}}\mathbf{A}^{-1}}{1+\bar{\mathbf{v}}^{\mathsf{T}}\mathbf{A}^{-1}\bar{\mathbf{u}}}$, which updates a known inverse after a rank-one change in $O(n^2)$ operations instead of $O(n^3)$.

AIML & 6G Connection

The WMI is behind the recursive least squares (RLS) adaptive filter and the Kalman filter, which update an inverse covariance as each new observation arrives. In a MIMO receiver,

Identity 2 shows that the MMSE filter can be computed either way: $\left(\sigma^2\mathbf{I}+\mathbf{H}^{\mathsf{H}}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathsf{H}} = \mathbf{H}^{\mathsf{H}}\left(\sigma^2\mathbf{I}+\mathbf{H}\mathbf{H}^{\mathsf{H}}\right)^{-1}$. One chooses whichever matrix to invert is smaller, the number of transmit or of receive antennas. The same trick speeds up Gaussian-process regression and Bayesian linear models in machine learning.

Chapter summary

- WMI: $(\mathbf{A}+\mathbf{U}\mathbf{C}\mathbf{V})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}(\mathbf{C}^{-1}+\mathbf{V}\mathbf{A}^{-1}\mathbf{U})^{-1}\mathbf{V}\mathbf{A}^{-1}$.
- Identity 1: $(\mathbf{I}+\mathbf{P})^{-1} = \mathbf{I} - (\mathbf{I}+\mathbf{P})^{-1}\mathbf{P}$.
- Identity 2: $(\mathbf{I}+\mathbf{P}\mathbf{Q})^{-1}\mathbf{P} = \mathbf{P}(\mathbf{I}+\mathbf{Q}\mathbf{P})^{-1}$.
- Rank-one update: $(\mathbf{I}+\bar{\mathbf{x}}\bar{\mathbf{x}}^{\mathsf{T}})^{-1} = \mathbf{I} - \dfrac{\bar{\mathbf{x}}\bar{\mathbf{x}}^{\mathsf{T}}}{1+\|\bar{\mathbf{x}}\|^2}$.

Exercises

13.1 Derive the Sherman–Morrison formula from the WMI.

13.2 Show that $(\mathbf{I}+c\,\bar{\mathbf{x}}\bar{\mathbf{x}}^{\mathsf{T}})^{-1} = \mathbf{I} - \dfrac{c\,\bar{\mathbf{x}}\bar{\mathbf{x}}^{\mathsf{T}}}{1+c\|\bar{\mathbf{x}}\|^2}$ for any scalar c with $1+c\|\bar{\mathbf{x}}\|^2 \neq 0$.

13.3 Use Theorem 13.4 to invert $\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$, and check your answer.

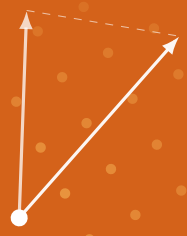
13.4 Verify Identity 1 for $\mathbf{P} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$.

13.5 Use Identity 2 to show that $\left(\sigma^2\mathbf{I}+\mathbf{H}^{\mathsf{H}}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathsf{H}} = \mathbf{H}^{\mathsf{H}}\left(\sigma^2\mathbf{I}+\mathbf{H}\mathbf{H}^{\mathsf{H}}\right)^{-1}$.

14

Linear Algebra and Probability

Gaussian random vectors, covariance matrices, i.i.d. samples and linear transformations



In this chapter

14.1 Gaussian random vectors 155
14.2 i.i.d. random variables 157
14.3 Linear transformations 157

We now explore applications of linear algebra in probability and statistics. Random vectors are described by a mean vector and a covariance matrix, and linear operations on them become matrix operations.

14.1 Gaussian random vectors

Definition 14.1 (Gaussian random variable)**.** Recall that the probability density function (PDF) of a Gaussian random variable X is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \tag{14.1}$$

Its mean and variance are

$$\mathbb{E}\{X\} = \mu, \qquad \mathbb{E}\{(X-\mu)^2\} = \sigma^2.$$

Definition 14.2 (Multivariate Gaussian)**.** Recall that the PDF of an n-dimensional Gaussian random vector $\bar{\mathbf{x}}$ is

$$f_{\bar{\mathbf{x}}}(\bar{\mathbf{x}}) = \frac{1}{\sqrt{(2\pi)^n |\mathbf{R}|}} e^{-\frac{1}{2}(\bar{\mathbf{x}}-\bar{=})^{\mathsf{T}}\mathbf{R}^{-1}(\bar{\mathbf{x}}-\bar{=})}. \tag{14.2}$$

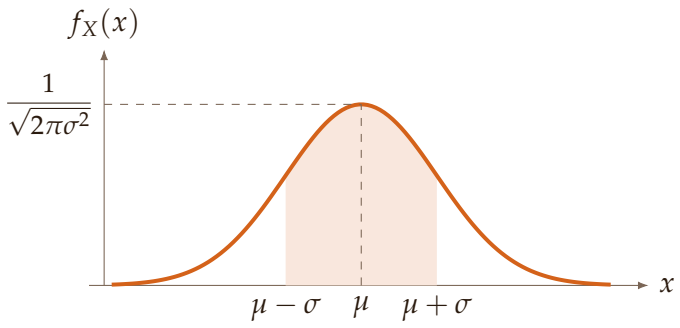


Figure 14.1 The Gaussian PDF: a bell-shaped curve centred at μ with peak value $1/\sqrt{2\pi\sigma^2}$. About 68% of the probability lies within $\mu \pm \sigma$ (shaded).

Its **mean vector** and **covariance matrix** are

$$\mathbb{E}\{\bar{\mathbf{x}}\} = \bar{=}, \qquad \mathbb{E}\left\{(\bar{\mathbf{x}} - \bar{=})(\bar{\mathbf{x}} - \bar{=})^{\mathsf{T}}\right\} = \mathbf{R}.$$

The exponent is a quadratic form in $\bar{\mathbf{x}} - \bar{=}$, and the normalization uses the determinant $|\mathbf{R}|$. For $n = 1$, $\mathbf{R} = \sigma^2$ and (14.2) reduces to (14.1).

Example 14.1 (A bivariate Gaussian PDF)

Determine the multivariate Gaussian PDF with mean and covariance matrix

$$\bar{=} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \qquad \mathbf{R} = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}.$$

Solution. We begin by evaluating $|\mathbf{R}|$ and $\mathbf{R}^{-1}$:

$$|\mathbf{R}| = \begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} = 8 - 1 = 7, \qquad \mathbf{R}^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}.$$

The multivariate Gaussian PDF is then

$$\begin{aligned} f(\bar{\mathbf{x}}) &= \frac{1}{\sqrt{(2\pi)^n |\mathbf{R}|}} e^{-\frac{1}{2}(\bar{\mathbf{x}}-\bar{=})^{\mathsf{T}}\mathbf{R}^{-1}(\bar{\mathbf{x}}-\bar{=})} \\ &= \frac{1}{\sqrt{(2\pi)^2 \times 7}} \exp\left(-\frac{1}{14}\begin{bmatrix} x_1 - 2 & x_2 + 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 - 2 \\ x_2 + 1 \end{bmatrix}\right). \end{aligned}$$

The quadratic form in the exponent simplifies as

$$\begin{aligned} &\begin{bmatrix} x_1 - 2 & x_2 + 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 - 2 \\ x_2 + 1 \end{bmatrix} \\ &\quad = 2(x_1 - 2)^2 + 4(x_2 + 1)^2 - 2(x_1 - 2)(x_2 + 1) \\ &\quad = 2x_1^2 + 4x_2^2 - 10x_1 + 12x_2 - 2x_1x_2 + 16. \end{aligned}$$

The multivariate Gaussian PDF is therefore

$$f(\bar{\mathbf{x}}) = \frac{1}{\sqrt{(2\pi)^2 \times 7}} e^{-\frac{1}{14}\left(2x_1^2 + 4x_2^2 - 10x_1 + 12x_2 - 2x_1x_2 + 16\right)}.$$

◀

14.2 i.i.d. random variables

Consider independent and identically distributed (**i.i.d.**) random variables $x_1, x_2, \ldots, x_n$ of mean 0 and variance σ^2. Therefore

$$\mathbb{E}\{x_i\} = 0, \qquad \mathbb{E}\{x_i x_j\} = \begin{cases} \sigma^2, & i = j, \\ 0, & i \neq j. \end{cases}$$

Let $\bar{\mathbf{x}} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^\mathsf{T}$. Then the mean vector is

$$\mathbb{E}\{\bar{\mathbf{x}}\} = \begin{bmatrix} \mathbb{E}\{x_1\} \\ \mathbb{E}\{x_2\} \\ \vdots \\ \mathbb{E}\{x_n\} \end{bmatrix} = \mathbf{0},$$

and the covariance matrix is

$$\mathbb{E}\{\bar{\mathbf{x}}\bar{\mathbf{x}}^\mathsf{T}\} = \mathbb{E}\left\{ \begin{bmatrix} x_1^2 & x_1x_2 & \cdots & x_1x_n \\ x_2x_1 & x_2^2 & \cdots & x_2x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_nx_1 & x_nx_2 & \cdots & x_n^2 \end{bmatrix} \right\} = \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{bmatrix} = \sigma^2\mathbf{I}.$$

Theorem 14.3 (Covariance of i.i.d. samples)

A vector of i.i.d. zero-mean samples of variance σ^2 has mean $\mathbf{0}$ and covariance

$$\mathbb{E}\{\bar{\mathbf{x}}\bar{\mathbf{x}}^\mathsf{T}\} = \sigma^2\mathbf{I}. \tag{14.3}$$

14.3 Linear transformations

Theorem 14.4 (Mean and covariance of $\mathbf{A}\bar{\mathbf{x}} + \bar{\mathbf{b}}$)

Let $\bar{\mathbf{y}} = \mathbf{A}\bar{\mathbf{x}} + \bar{\mathbf{b}}$, where $\bar{\mathbf{x}}$ has mean $\bar{}$ and covariance $\mathbf{R}$. Then $\bar{\mathbf{y}}$ has mean and covariance

$$\bar{}_y = \mathbf{A}\bar{} + \bar{\mathbf{b}}, \qquad \mathbf{R}_y = \mathbf{A}\mathbf{R}\mathbf{A}^\mathsf{T}. \tag{14.4}$$

If $\bar{\mathbf{x}}$ is Gaussian, then any linear transformation $\bar{\mathbf{y}} = \mathbf{A}\bar{\mathbf{x}} + \bar{\mathbf{b}}$ is also Gaussian.

Proof. By linearity of expectation, the mean of $\bar{\mathbf{y}}$ is

$$\mathbb{E}\{\bar{\mathbf{y}}\} = \mathbb{E}\{\mathbf{A}\bar{\mathbf{x}} + \bar{\mathbf{b}}\} = \mathbf{A}\,\mathbb{E}\{\bar{\mathbf{x}}\} + \bar{\mathbf{b}} = \mathbf{A}\bar{} + \bar{\mathbf{b}} = \bar{}_y.$$

Note that

$$\bar{\mathbf{y}} - \bar{}_y = \mathbf{A}\bar{\mathbf{x}} + \bar{\mathbf{b}} - \mathbf{A}\bar{} - \bar{\mathbf{b}} = \mathbf{A}(\bar{\mathbf{x}} - \bar{}).$$

The covariance matrix is therefore

$$\mathbb{E}\left\{(\bar{\mathbf{y}}-\bar{=}_y)(\bar{\mathbf{y}}-\bar{=}_y)^{\mathsf{T}}\right\} = \mathbb{E}\left\{\mathbf{A}(\bar{\mathbf{x}}-\bar{=})\left(\mathbf{A}(\bar{\mathbf{x}}-\bar{=})\right)^{\mathsf{T}}\right\}$$
$$= \mathbf{A}\,\mathbb{E}\left\{(\bar{\mathbf{x}}-\bar{=})(\bar{\mathbf{x}}-\bar{=})^{\mathsf{T}}\right\}\mathbf{A}^{\mathsf{T}} = \mathbf{A}\mathbf{R}\mathbf{A}^{\mathsf{T}}.$$

■

Example 14.2 (Transforming a Gaussian vector)

Let $\bar{\mathbf{x}}$ be a 2-dimensional Gaussian random vector with mean and covariance

$$\bar{=} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \qquad \mathbf{R} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix},$$

and consider

$$\bar{\mathbf{y}} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\bar{\mathbf{x}} + \begin{bmatrix} -2 \\ 4 \end{bmatrix}.$$

Find the mean and covariance of $\bar{\mathbf{y}}$.

Solution. The mean is

$$\bar{=}_y = \mathbf{A}\bar{=} + \bar{\mathbf{b}} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}.$$

The covariance matrix is

$$\mathbf{R}_y = \mathbf{A}\mathbf{R}\mathbf{A}^{\mathsf{T}} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & -3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 3 & 12 \end{bmatrix}.$$

Since $\bar{\mathbf{x}}$ is Gaussian, $\bar{\mathbf{y}}$ is Gaussian with this mean and covariance. ◂

14.3.1 Special case: a linear combination of i.i.d. variables

We now explore the special case of a linear transformation of i.i.d. random variables. Consider i.i.d. random variables x_i with mean 0 and variance σ^2, and let

$$y = \bar{\mathbf{a}}^{\mathsf{T}}\bar{\mathbf{x}}.$$

Then $\mathbb{E}\{y\} = \bar{\mathbf{a}}^{\mathsf{T}}\mathbb{E}\{\bar{\mathbf{x}}\} = 0$, and

$$\mathbb{E}\{y^2\} = \mathbb{E}\left\{(\bar{\mathbf{a}}^{\mathsf{T}}\bar{\mathbf{x}})^2\right\} = \mathbb{E}\left\{(\bar{\mathbf{a}}^{\mathsf{T}}\bar{\mathbf{x}})(\bar{\mathbf{a}}^{\mathsf{T}}\bar{\mathbf{x}})^{\mathsf{T}}\right\}$$
$$= \mathbb{E}\left\{\bar{\mathbf{a}}^{\mathsf{T}}\bar{\mathbf{x}}\bar{\mathbf{x}}^{\mathsf{T}}\bar{\mathbf{a}}\right\} = \bar{\mathbf{a}}^{\mathsf{T}}\mathbb{E}\{\bar{\mathbf{x}}\bar{\mathbf{x}}^{\mathsf{T}}\}\bar{\mathbf{a}}.$$

Using Theorem 14.3, this simplifies to

$$\mathbb{E}\{y^2\} = \bar{\mathbf{a}}^{\mathsf{T}}\sigma^2\mathbf{I}\bar{\mathbf{a}} = \sigma^2\bar{\mathbf{a}}^{\mathsf{T}}\bar{\mathbf{a}} = \sigma^2\|\bar{\mathbf{a}}\|^2.$$

Theorem 14.5 (Variance of a linear combination)

If $\bar{\mathbf{x}}$ has i.i.d. zero-mean entries of variance σ^2, then $y = \bar{\mathbf{a}}^{\mathsf{T}}\bar{\mathbf{x}}$ has mean 0 and variance

$$\mathbb{E}\{y^2\} = \sigma^2 \|\bar{\mathbf{a}}\|^2. \tag{14.5}$$

AIML & 6G Connection

Receiver noise in wireless systems is modelled as i.i.d. Gaussian, with covariance $\sigma^2\mathbf{I}$. After a linear receive filter $\bar{\mathbf{w}}$, the noise $\bar{\mathbf{w}}^{\mathsf{H}}\bar{\mathbf{n}}$ has power $\sigma^2\|\bar{\mathbf{w}}\|^2$ by Theorem 14.5, and this sets the SNR of a beamformer. After a matrix filter $\mathbf{W}$, the noise covariance is $\mathbf{W}\mathbf{R}\mathbf{W}^{\mathsf{H}}$. In machine learning, the same rule $\mathbf{A}\mathbf{R}\mathbf{A}^{\mathsf{T}}$ explains PCA and whitening: with $\mathbf{R} = \mathbf{V}\blacksquare\mathbf{V}^{\mathsf{T}}$, the transform $\blacksquare^{-1/2}\mathbf{V}^{\mathsf{T}}(\bar{\mathbf{x}} - \bar{=})$ has covariance $\mathbf{I}$.

Chapter summary

- A Gaussian random vector has PDF $\frac{1}{\sqrt{(2\pi)^n|\mathbf{R}|}}\, e^{-\frac{1}{2}(\bar{\mathbf{x}}-\bar{=})^{\mathsf{T}}\mathbf{R}^{-1}(\bar{\mathbf{x}}-\bar{=})}$.
- Its mean is $\bar{=}$ and its covariance matrix is $\mathbf{R} = \mathbb{E}\{(\bar{\mathbf{x}} - \bar{=})(\bar{\mathbf{x}} - \bar{=})^{\mathsf{T}}\}$.
- i.i.d. zero-mean samples of variance σ^2 have covariance $\sigma^2\mathbf{I}$.
- $\bar{\mathbf{y}} = \mathbf{A}\bar{\mathbf{x}} + \bar{\mathbf{b}}$ has mean $\mathbf{A}\bar{=} + \bar{\mathbf{b}}$ and covariance $\mathbf{A}\mathbf{R}\mathbf{A}^{\mathsf{T}}$; it stays Gaussian.
- For i.i.d. $\bar{\mathbf{x}}$, $y = \bar{\mathbf{a}}^{\mathsf{T}}\bar{\mathbf{x}}$ has variance $\sigma^2\|\bar{\mathbf{a}}\|^2$.

Exercises

14.1 Write the PDF of the Gaussian vector with $\bar{=} = \mathbf{0}$ and $\mathbf{R} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, expanding the quadratic form.

14.2 Let $\bar{\mathbf{x}}$ have i.i.d. zero-mean entries of variance σ^2, and $\bar{\mathbf{y}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\bar{\mathbf{x}}$. Find the covariance of $\bar{\mathbf{y}}$. Are y_1 and y_2 uncorrelated?

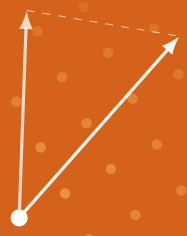
14.3 Show that every covariance matrix is symmetric and positive semidefinite.

14.4 Let $\mathbf{R} = \mathbf{V}\blacksquare\mathbf{V}^{\mathsf{T}}$ with all $\lambda_i > 0$. Show that $\bar{\mathbf{z}} = \blacksquare^{-1/2}\mathbf{V}^{\mathsf{T}}(\bar{\mathbf{x}} - \bar{=})$ has mean $\mathbf{0}$ and covariance $\mathbf{I}$.

14.5 Find the variance of the sample sum $x_1 + x_2 + \cdots + x_n$ and of the sample mean $\frac{1}{n}(x_1 + \cdots + x_n)$ of i.i.d. variables with variance σ^2.

15

Cayley–Hamilton Theorem



Every matrix satisfies its characteristic equation: proof, inverses and matrix polynomials

In this chapter

15.1 The Cayley–Hamilton theorem 161
15.2 Proof for a diagonalizable matrix 161
15.3 Examples 162

15.1 The Cayley–Hamilton theorem

Consider the characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$,

$$p(\lambda) = |\mathbf{A} - \lambda \mathbf{I}| = a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n.$$

Let λ_i denote the eigenvalues of $\mathbf{A}$. We know that they are the roots of the characteristic polynomial (Chapter 8):

$$p(\lambda_i) = a_0 + a_1\lambda_i + a_2\lambda_i^2 + \cdots + a_n\lambda_i^n = 0.$$

The Cayley–Hamilton theorem says that the same polynomial also vanishes when the scalar λ is replaced by the matrix $\mathbf{A}$ itself.

Theorem 15.1 (Cayley–Hamilton)

Every square matrix satisfies its own characteristic equation:

$$p(\mathbf{A}) = a_0\mathbf{I} + a_1\mathbf{A} + a_2\mathbf{A}^2 + \cdots + a_n\mathbf{A}^n = \mathbf{0}. \tag{15.1}$$

Note that the constant term a_0 becomes $a_0\mathbf{I}$, since $\mathbf{A}^0 = \mathbf{I}$.

15.2 Proof for a diagonalizable matrix

Proof of Theorem 15.1 for diagonalizable matrices. For simplicity, consider a diagonalizable matrix $\mathbf{A} = \mathbf{U}\blacksquare\mathbf{U}^{-1}$ (Chapter 8). It is easy to see that

$$\mathbf{A}^n = \mathbf{U}\blacksquare\mathbf{U}^{-1} \times \mathbf{U}\blacksquare\mathbf{U}^{-1} \times \cdots \times \mathbf{U}\blacksquare\mathbf{U}^{-1} = \mathbf{U}\blacksquare^n\mathbf{U}^{-1},$$

since each inner product $\mathbf{U}^{-1}\mathbf{U} = \mathbf{I}$ cancels. It therefore follows that

$$\begin{aligned} p(\mathbf{A}) &= a_0\mathbf{U}\mathbf{U}^{-1} + a_1\mathbf{U}\blacksquare\mathbf{U}^{-1} + a_2\mathbf{U}\blacksquare^2\mathbf{U}^{-1} + \cdots + a_n\mathbf{U}\blacksquare^n\mathbf{U}^{-1} \\ &= \mathbf{U}\left(a_0\blacksquare^0 + a_1\blacksquare + \cdots + a_n\blacksquare^n\right)\mathbf{U}^{-1}. \end{aligned}$$

Consider now the matrix $a_0\blacksquare^0 + a_1\blacksquare + \cdots + a_n\blacksquare^n$. It is diagonal, and its ith diagonal element is

$$a_0\lambda_i^0 + a_1\lambda_i + a_2\lambda_i^2 + \cdots + a_n\lambda_i^n = p(\lambda_i) = 0.$$

Therefore

$$\begin{aligned} a_0\blacksquare^0 + a_1\blacksquare + \cdots + a_n\blacksquare^n = \mathbf{0} \;&\Rightarrow\; \mathbf{U}\left(a_0\blacksquare^0 + a_1\blacksquare + \cdots + a_n\blacksquare^n\right)\mathbf{U}^{-1} = \mathbf{0} \\ &\Rightarrow\; p(\mathbf{A}) = \mathbf{0}. \end{aligned}$$

■

In compact form, $p(\mathbf{A}) = \mathbf{U}\,\mathrm{diag}\left(p(\lambda_1), \ldots, p(\lambda_n)\right)\mathbf{U}^{-1} = \mathbf{0}$.

> **Remark.** The theorem holds for every square matrix, diagonalizable or not. A general proof uses the Jordan normal form $\mathbf{A} = \mathbf{P}\mathbf{J}\mathbf{P}^{-1}$ (Chapter 9): $p(\mathbf{J})$ is block diagonal, and each Jordan block $\mathbf{J}_i$ of size k is annihilated because $p(\lambda)$ contains the factor $(\lambda - \lambda_i)^k$.

15.3 Examples

Example 15.1 (Verifying the theorem)

Verify the Cayley–Hamilton theorem for $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$.

Solution. The characteristic equation is

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 12 \;\Rightarrow\; p(\lambda) = \lambda^2 - 3\lambda - 10.$$

Now check:

$$\begin{aligned} p(\mathbf{A}) &= \mathbf{A}^2 - 3\mathbf{A} - 10\mathbf{I} \\ &= \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} - 3\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} - 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 16 & 9 \\ 12 & 13 \end{bmatrix} - \begin{bmatrix} 6 & 9 \\ 12 & 3 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = \mathbf{0}. \end{aligned}$$

◀

Example 15.2 (The inverse from Cayley–Hamilton)

Compute $\mathbf{A}^{-1}$ for the matrix of Example 15.1 using the theorem.

Solution. From $p(\mathbf{A}) = \mathbf{0}$,

$$\mathbf{A}^2 - 3\mathbf{A} - 10\mathbf{I} = \mathbf{0} \;\Rightarrow\; \mathbf{A}^2 - 3\mathbf{A} = 10\mathbf{I} \;\Rightarrow\; \mathbf{A}^{-1}(\mathbf{A}^2 - 3\mathbf{A}) = 10\mathbf{A}^{-1}$$
$$\Rightarrow\; \mathbf{A} - 3\mathbf{I} = 10\mathbf{A}^{-1} \;\Rightarrow\; \mathbf{A}^{-1} = \tfrac{1}{10}(\mathbf{A} - 3\mathbf{I}).$$

Continuing,

$$\mathbf{A}^{-1} = \frac{1}{10}\left(\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} - 3\mathbf{I}\right) = \frac{1}{10}\begin{bmatrix} -1 & 3 \\ 4 & -2 \end{bmatrix}.$$

◀

Remark. In general, if $a_0 = p(0) = |\mathbf{A}| \neq 0$, then $p(\mathbf{A}) = \mathbf{0}$ gives $\mathbf{A}^{-1} = -\frac{1}{a_0}\left(a_1\mathbf{I} + a_2\mathbf{A} + \cdots + a_n\mathbf{A}^{n-1}\right)$: the inverse of a matrix is a polynomial in the matrix.

Example 15.3 (A matrix polynomial)

For the matrix of Example 15.1, evaluate

$$f(\mathbf{A}) = \mathbf{A}^5 - 3\mathbf{A}^4 - 9\mathbf{A}^3 - 5\mathbf{A}^2 - 2\mathbf{A} + 23\mathbf{I}.$$

Solution. Consider the polynomial $f(\lambda) = \lambda^5 - 3\lambda^4 - 9\lambda^3 - 5\lambda^2 - 2\lambda + 23$. Dividing it by $p(\lambda) = \lambda^2 - 3\lambda - 10$ gives

$$f(\lambda) = (\lambda^2 - 3\lambda - 10)(\lambda^3 + \lambda - 2) + 2\lambda + 3 = p(\lambda)(\lambda^3 + \lambda - 2) + 2\lambda + 3.$$

Therefore one can write $f(\mathbf{A})$ as

$$f(\mathbf{A}) = \underbrace{(\mathbf{A}^2 - 3\mathbf{A} - 10\mathbf{I})}_{p(\mathbf{A}) = \mathbf{0}}(\mathbf{A}^3 + \mathbf{A} - 2\mathbf{I}) + 2\mathbf{A} + 3\mathbf{I} = 2\mathbf{A} + 3\mathbf{I},$$

so

$$f(\mathbf{A}) = 2\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} + 3\mathbf{I} = \begin{bmatrix} 7 & 6 \\ 8 & 5 \end{bmatrix}.$$

◀

In general, any polynomial $f(\mathbf{A})$ reduces to its remainder after dividing $f(\lambda)$ by $p(\lambda)$, a polynomial of degree less than n.

AIML & 6G Connection

Cayley–Hamilton means that every power $\mathbf{A}^k$, and so every analytic function of $\mathbf{A}$ such as the matrix exponential $e^{\mathbf{A}t}$, can be written as a polynomial of degree at most $n-1$ in $\mathbf{A}$. This is used to solve the linear state equations $\dot{\mathbf{x}} = \mathbf{A}\bar{\mathbf{x}}$ of control systems and to analyze linear recurrent networks and state-space sequence models in machine learning. It also explains why Krylov subspace methods, which build solutions from $\bar{\mathbf{b}}, \mathbf{A}\bar{\mathbf{b}}, \mathbf{A}^2\bar{\mathbf{b}}, \ldots$, need at most n steps.

Chapter summary

- The characteristic polynomial is $p(\lambda) = |\mathbf{A} - \lambda \mathbf{I}|$.
- Cayley–Hamilton: every matrix satisfies $p(\mathbf{A}) = \mathbf{0}$.
- For diagonalizable $\mathbf{A}$, $p(\mathbf{A}) = \mathbf{U}\,\mathrm{diag}(p(\lambda_i))\mathbf{U}^{-1} = \mathbf{0}$.
- The theorem gives $\mathbf{A}^{-1}$ as a polynomial in $\mathbf{A}$.
- Any polynomial $f(\mathbf{A})$ reduces to its remainder after dividing $f(\lambda)$ by $p(\lambda)$.

Exercises

15.1 Verify the Cayley–Hamilton theorem for $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and use it to find the inverse.

15.2 For $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$, use $\mathbf{A}^2 = 3\mathbf{A} + 10\mathbf{I}$ to compute $\mathbf{A}^3$ without multiplying three matrices.

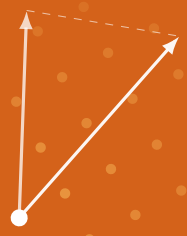
15.3 Show that every 2×2 matrix satisfies $\mathbf{A}^2 - \mathrm{tr}(\mathbf{A})\,\mathbf{A} + \det(\mathbf{A})\,\mathbf{I} = \mathbf{0}$.

15.4 Verify the theorem directly for the Jordan block $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, which is not diagonalizable.

15.5 Use the theorem to show that $\mathbf{A}$ is invertible if and only if $p(0) \neq 0$.

16

Discrete-Time Markov Chains



Discrete-time Markov chains, transition matrices, n-step transitions and limiting distributions

In this chapter

16.1 Discrete-time Markov chains 165
16.2 The transition matrix 166
16.3 n-step transitions 167
16.4 Limiting distribution 168

16.1 Discrete-time Markov chains

Consider a stochastic model of a system that transitions between states at specific, discrete time steps. Let X_n be the state of the system at time n, for $n = 0, 1, 2, \ldots$. Suppose we are currently at time $n = 10$, so we have observed $X_0, X_1, \ldots, X_{10}$. Can we predict, in a probabilistic way, the state of the system at time $n = 11$? This question brings us to the concept of a Markov chain.

Definition 16.1 (Discrete-time Markov chain). A stochastic process $\{X_n, n \geq 0\}$ on a state space S is a **discrete-time Markov chain** (DTMC) if

$$P\left(X_{n+1} = j \mid X_n = i, X_{n-1}, \ldots, X_0\right) = P\left(X_{n+1} = j \mid X_n = i\right). \tag{16.1}$$

This is the **Markov property**: given the present state of the system, namely X_n, the future state of the DTMC, namely X_{n+1}, is independent of its past, namely $X_0, X_1, \ldots, X_{n-1}$.

Definition 16.2 (Time homogeneity and transition probability). A DTMC is **time homogeneous** if, for all n,

$$P(X_{n+1} = j \mid X_n = i) = P(X_1 = j \mid X_0 = i).$$

This quantity is called a **one-step transition probability** of the DTMC and is denoted p_{ij}.

16.2 The transition matrix

Definition 16.3 (Transition matrix). It is convenient to arrange the p_{ij} of an N-state DTMC in an $N \times N$ matrix,

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1N} \\ p_{21} & p_{22} & \cdots & p_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ p_{N1} & p_{N2} & \cdots & p_{NN} \end{bmatrix}, \tag{16.2}$$

called the **one-step transition probability matrix**, or **transition matrix** for short. The rows correspond to the starting state and the columns to the ending state of a transition.

Since from state i the chain must move to some state, each row of $\mathbf{P}$ sums to one:

$$\sum_{j=1}^{N} p_{ij} = 1.$$

Example 16.1 (A three-state DTMC)

Consider a DTMC with transition matrix

$$\mathbf{P} = \begin{bmatrix} 0.30 & 0.20 & 0.50 \\ 0.40 & 0.00 & 0.60 \\ 0.35 & 0.00 & 0.65 \end{bmatrix}.$$

Its transition diagram is shown in Figure 16.1. The rows satisfy $\sum_j p_{ij} = 1$; for example, row 1 gives $0.30 + 0.20 + 0.50 = 1$.

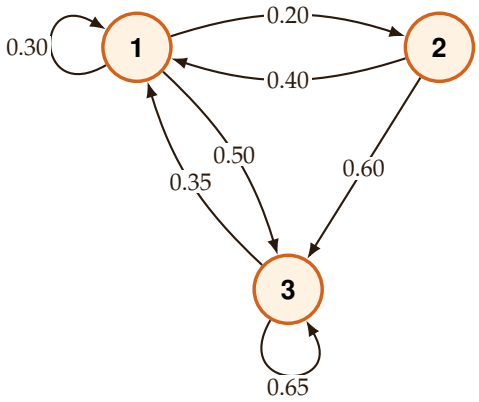


Figure 16.1 Transition diagram of the DTMC of Example 16.1. Each arrow $i \to j$ carries the probability p_{ij}; the arrows leaving each state sum to one.

Example 16.2 (A machine that is up or down)

A company uses a machine that is either up or down. If it is up at the beginning of a day, it is up at the beginning of the next day with probability 0.95, or it fails with probability 0.05. If it is down at the beginning of a day, it is down at the beginning of the next day

with probability 0.10, or the repair is completed and it is up with probability 0.90. Find the transition matrix.

Solution. With state 0 = down and state 1 = up, the transition probability matrix of the DTMC is

$$\mathbf{P} = \begin{bmatrix} 0.10 & 0.90 \\ 0.05 & 0.95 \end{bmatrix},$$

with rows and columns ordered (down, up). ◂



Figure 16.2 Transition diagram of the machine of Example 16.2.

16.3 n-step transitions

Theorem 16.4 (n-step transition matrix)

The n-step transition probability matrix $\mathbf{P}^{(n)}$, whose entries are the conditional probabilities $P(X_n = j \mid X_0 = i)$, is the nth power of the transition matrix:

$$\mathbf{P}^{(n)} = \mathbf{P}^n. \tag{16.3}$$

Proof. To go from i to j in two steps, the chain passes through some intermediate state k:

$$\begin{aligned} P(X_2 = j \mid X_0 = i) &= \sum_{k=1}^{N} P(X_1 = k \mid X_0 = i)\, P(X_2 = j \mid X_1 = k) \\ &= \sum_{k=1}^{N} p_{ik} p_{kj} = [\mathbf{P}^2]_{ij}, \end{aligned}$$

using the Markov property and time homogeneity. Repeating the argument gives $\mathbf{P}^{(n)} = \mathbf{P}^n$. ■

Example 16.3 (Two and ten steps)

For the machine of Example 16.2, compute the 2- and 10-step transition probability matrices.

Solution. With $\mathbf{P} = \begin{bmatrix} 0.10 & 0.90 \\ 0.05 & 0.95 \end{bmatrix}$, these are

$$\mathbf{P}^2 = \begin{bmatrix} 0.0550 & 0.9450 \\ 0.0525 & 0.9475 \end{bmatrix}, \qquad \mathbf{P}^{10} = \begin{bmatrix} 0.05263158 & 0.94736842 \\ 0.05263158 & 0.94736842 \end{bmatrix}.$$

For instance, $[\mathbf{P}^2]_{11} = 0.10 \times 0.10 + 0.90 \times 0.05 = 0.0550$. ◂

16.4 Limiting distribution

Examine the powers of the machine's transition matrix:

$$\mathbf{P}^{10} = \begin{bmatrix} 0.05263158 & 0.94736842 \\ 0.05263158 & 0.94736842 \end{bmatrix}, \qquad \mathbf{P}^{100} = \begin{bmatrix} 0.05263158 & 0.94736842 \\ 0.05263158 & 0.94736842 \end{bmatrix}.$$

Observe that $\mathbf{P}^{10} \approx \mathbf{P}^{100}$. This is explained by the limiting distribution.

Definition 16.5 (Limiting distribution). If the probability mass function (PMF) of X_n approaches a limit as $n \to \infty$,

$$\pi_j = \lim_{n\to\infty} P(X_n = j),$$

the row vector

$$\bar{\boldsymbol{\pi}} = \begin{bmatrix} \pi_1 & \pi_2 & \cdots & \pi_N \end{bmatrix}$$

is called the **limiting** or **steady-state distribution**.

Theorem 16.6 (Balance equation)

If a limiting distribution exists, it is a solution of

$$\bar{\boldsymbol{\pi}} = \bar{\boldsymbol{\pi}}\mathbf{P}, \qquad \sum_{j=1}^{N} \pi_j = 1. \tag{16.4}$$

Proof. By the law of total probability,

$$P(X_{n+1} = j) = \sum_{i=1}^{N} P(X_n = i)\, p_{ij}.$$

Taking the limit $n \to \infty$ on both sides,

$$\lim_{n\to\infty} P(X_{n+1} = j) = \lim_{n\to\infty} \sum_{i=1}^{N} P(X_n = i)\, p_{ij} \;\Rightarrow\; \pi_j = \sum_{i=1}^{N} \pi_i p_{ij} \;\Rightarrow\; \bar{\boldsymbol{\pi}} = \bar{\boldsymbol{\pi}}\mathbf{P}.$$

■

In the language of Chapter 8, $\bar{\boldsymbol{\pi}}^{\mathsf{T}}$ is an eigenvector of $\mathbf{P}^{\mathsf{T}}$ with eigenvalue 1, normalized so that its entries sum to one.

Example 16.4 (Limiting distribution from the powers of $\mathbf{P}$)

The limiting distribution can also be found from the powers of $\mathbf{P}$. Find it for the three-state DTMC of Example 16.1.

Solution. The n-step transition probability matrices for small values of n are

$$\mathbf{P}^2 = \begin{bmatrix} 0.3450 & 0.0600 & 0.5950 \\ 0.3300 & 0.0800 & 0.5900 \\ 0.3325 & 0.0700 & 0.5975 \end{bmatrix}, \qquad \mathbf{P}^3 = \begin{bmatrix} 0.3358 & 0.0690 & 0.5952 \\ 0.3375 & 0.0660 & 0.5965 \\ 0.3369 & 0.0665 & 0.5966 \end{bmatrix}.$$

For higher values of n,

$$\mathbf{P}^5 = \begin{bmatrix} 0.3365 & 0.0673 & 0.5961 \\ 0.3366 & 0.0673 & 0.5962 \\ 0.3365 & 0.0673 & 0.5962 \end{bmatrix},$$

$$\mathbf{P}^n \approx \begin{bmatrix} 0.3365 & 0.0673 & 0.5962 \\ 0.3365 & 0.0673 & 0.5962 \\ 0.3365 & 0.0673 & 0.5962 \end{bmatrix} \quad (n \geq 6).$$

All the rows of $\mathbf{P}^n$ become the same in the limit. This implies that the limiting distribution of X_n is the same regardless of the initial distribution:

$$\bar{\boldsymbol{\pi}} = \begin{bmatrix} 0.3365 & 0.0673 & 0.5962 \end{bmatrix} = \begin{bmatrix} \frac{35}{104} & \frac{7}{104} & \frac{62}{104} \end{bmatrix}.$$

Check. Solving $\bar{\boldsymbol{\pi}} = \bar{\boldsymbol{\pi}}\mathbf{P}$ exactly confirms this: $\pi_2 = 0.2\,\pi_1$ from the second column, and the first column gives $0.7\,\pi_1 = 0.4\,\pi_2 + 0.35\,\pi_3$, so $\pi_3 = \frac{62}{35}\pi_1$. Normalizing, $\pi_1 = \frac{35}{104}$. ◂

Definition 16.7 (Limiting transition matrix)**.** Let $\mathbf{P}^{(\infty)}$ be the n-step transition probability matrix as $n \to \infty$:

$$\mathbf{P}^{(\infty)} = \lim_{n\to\infty} \mathbf{P}^{(n)}.$$

The limiting distribution is given by each row of $\mathbf{P}^{(\infty)}$.

Example 16.5 (Steady state of the machine)

For the machine of Example 16.2, solve $\bar{\boldsymbol{\pi}} = \bar{\boldsymbol{\pi}}\mathbf{P}$ and compare with $\mathbf{P}^{10}$.

Solution. The first column of $\bar{\boldsymbol{\pi}} = \bar{\boldsymbol{\pi}}\mathbf{P}$ gives $\pi_0 = 0.10\,\pi_0 + 0.05\,\pi_1$, so $\pi_1 = 18\,\pi_0$. With $\pi_0 + \pi_1 = 1$,

$$\bar{\boldsymbol{\pi}} = \begin{bmatrix} \frac{1}{19} & \frac{18}{19} \end{bmatrix} = \begin{bmatrix} 0.05263158 & 0.94736842 \end{bmatrix},$$

exactly the rows of $\mathbf{P}^{10}$ in Example 16.3. In the long run, the machine is up about 94.7% of the days. ◂

AIML & 6G Connection

Google's PageRank models a web surfer as a DTMC on the pages of the web; the ranking is its limiting distribution $\bar{\boldsymbol{\pi}} = \bar{\boldsymbol{\pi}}\mathbf{P}$, computed by repeated multiplication (the power method). In wireless communication, the Gilbert–Elliott model describes a bursty fading channel as a two-state good/bad Markov chain, just like the machine example. Markov chains also underlie Markov decision processes in reinforcement learning, hidden Markov models, and MCMC sampling.

Chapter summary

- A DTMC satisfies the Markov property: the future depends only on the present state.
- It is time homogeneous if $P(X_{n+1} = j \mid X_n = i) = p_{ij}$ does not depend on n.
- Its transition matrix $\mathbf{P}$ has entries p_{ij}, with rows indexed by the starting state; each row sums to 1.
- The n-step transition matrix is $\mathbf{P}^{(n)} = \mathbf{P}^n$.
- The limiting distribution solves $\bar{\boldsymbol{\pi}} = \bar{\boldsymbol{\pi}}\mathbf{P}$ and equals each row of $\mathbf{P}^{(\infty)}$.

Exercises

16.1 Show that every row of $\mathbf{P}^n$ sums to 1.

16.2 Show that $\mathbf{P}\bar{\mathbf{1}} = \bar{\mathbf{1}}$, where $\bar{\mathbf{1}}$ is the all-ones vector, so 1 is always an eigenvalue of a transition matrix.

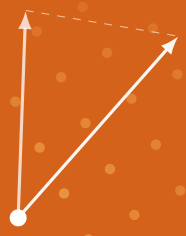
16.3 For the general two-state chain $\mathbf{P} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$ with $a + b > 0$, show that $\bar{\boldsymbol{\pi}} = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}$, and check it against the machine example.

16.4 Verify that $\bar{\boldsymbol{\pi}} = \frac{1}{104}\begin{bmatrix} 35 & 7 & 62 \end{bmatrix}$ satisfies $\bar{\boldsymbol{\pi}} = \bar{\boldsymbol{\pi}}\mathbf{P}$ for the chain of Example 16.1.

16.5 If the initial distribution is $\bar{\boldsymbol{\pi}}_0$ (a row vector), show that the distribution of X_n is $\bar{\boldsymbol{\pi}}_0\mathbf{P}^n$.

A

Notation



Symbol		Meaning
$\bar{\mathbf{u}}$	`\vect{u}`	Column vector
$\mathbf{A}$	`\mat{A}`	Matrix
$\mathbf{A}^{\mathsf{T}}$, $\mathbf{A}^{\mathsf{H}}$		Transpose, Hermitian (conjugate) transpose
$\|\bar{\mathbf{u}}\|$		Euclidean norm
$\langle \bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle$		Inner product
$\mathbb{R}^n$, $\mathbb{C}^n$		Real and complex n-dimensional spaces
$\mathrm{tr}(\mathbf{A})$, $\mathrm{rank}(\mathbf{A})$		Trace, rank
$\mathrm{span}\{\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_k\}$		Span of a set of vectors