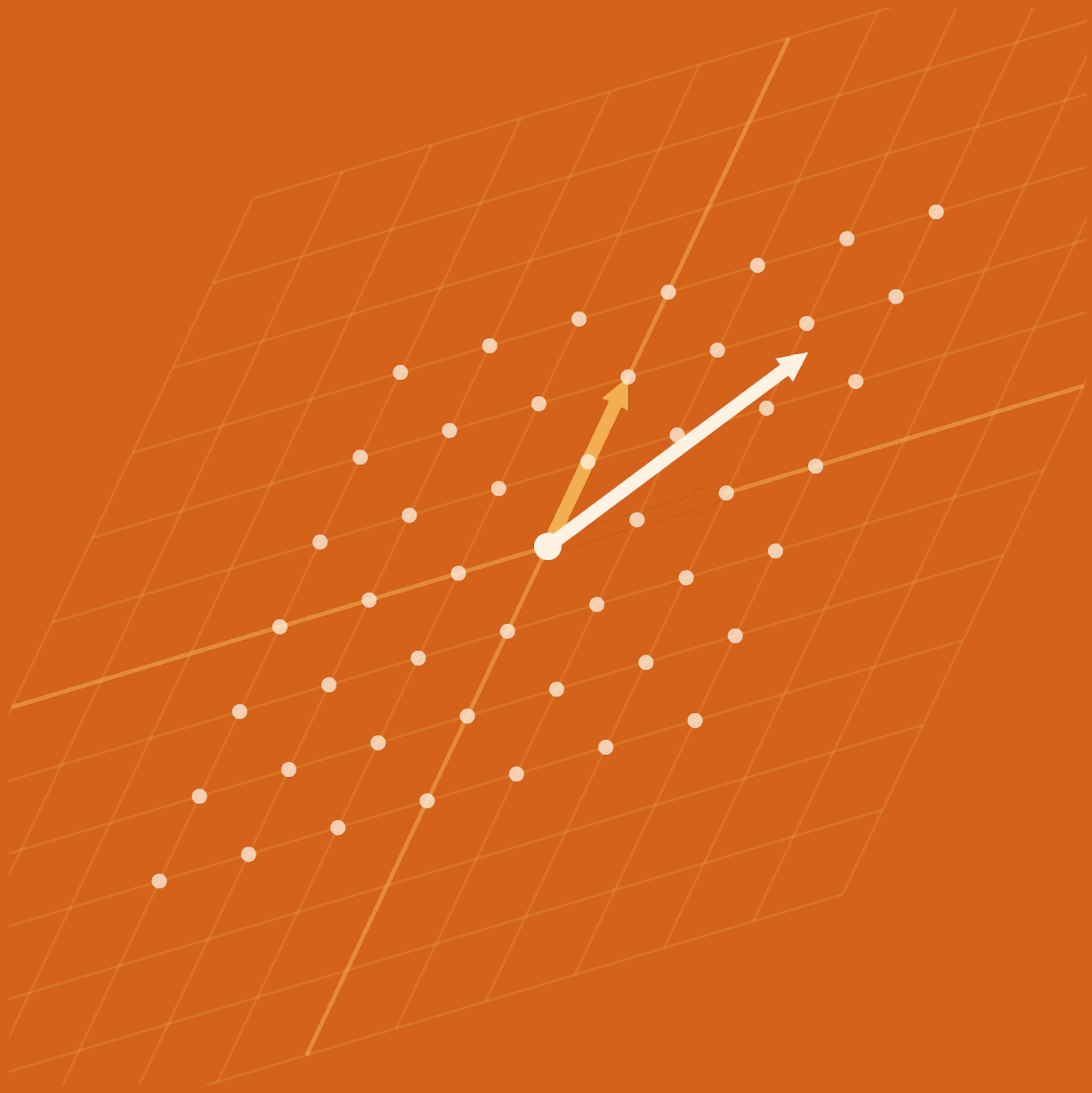
LINEAR ALGEBRA FOR AIML AND 6G

Discrete-Time Markov Chains

Lecture 26

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Outline

1. **Discrete-time Markov chains**
   The Markov property and time homogeneity
2. **Transition matrix**
   One-step transition probabilities and diagrams
3. **n-step transitions**
   Powers of the transition matrix
4. **Limiting distribution**
   The steady state $\pi = \pi P$

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Discrete-time Markov chains

The Markov property and time homogeneity



Stochastic model

- Consider a **stochastic model** of a system that transitions between states at specific, **discrete time** steps.
- Let X_n be the state of the system at time n for $n = 0,1,2,...$

Stochastic model

- Suppose we are currently at time $n = 10$.
  - That is, we have observed X_0, X_1, ..., X_{10}

Markov chain

- The question is: can we predict, in a probabilistic way, the state of the system at time $n = 11$?
  - This brings us to the concept of a **Markov chain**

Markov chain

- A stochastic process $X_n, n \geq 0$ on state space S is said to be a **discrete-time Markov chain (DTMC)** if

$$P\left(X_{n+1}=j \middle| X_n=i, X_{n-1}, \ldots, X_0\right) = P\left(X_{n+1}=j \middle| X_n=i\right)$$

Markov property

- Given the **present state** of the system (namely X_n),
  - the **future state** of the DTMC (namely X_{n+1}) is **independent of its past** (namely X_0, X_1,…,X_{n-1}).

Homogeneous

- A DTMC is said to be **time homogeneous** if, for all n

$$P\left(X_{n+1}=j|X_n=i\right)=P\left(X_1=j|X_0=i\right)$$

Transition probability

- The quantity below is called a **one-step transition probability** of the DTMC

$$P\left(X_{n+1}=j|X_n=i\right)=P\left(X_1=j|X_0=i\right)$$

- This is denoted by p_{ij}

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Transition matrix

One-step transition probabilities and diagrams



Transition matrix

- It is convenient to arrange the p_{ij} in an $N \times N$ matrix as shown below

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1N} \\ p_{21} & p_{22} & \cdots & p_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ p_{N1} & p_{N2} & \cdots & p_{NN} \end{bmatrix}$$

- This is the **one-step transition probability matrix**, or **transition matrix** for short

Transition matrix

- Note that the **rows** correspond to the starting state
  - and the **columns** correspond to the ending state of a transition

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1N} \\ p_{21} & p_{22} & \cdots & p_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ p_{N1} & p_{N2} & \cdots & p_{NN} \end{bmatrix}$$

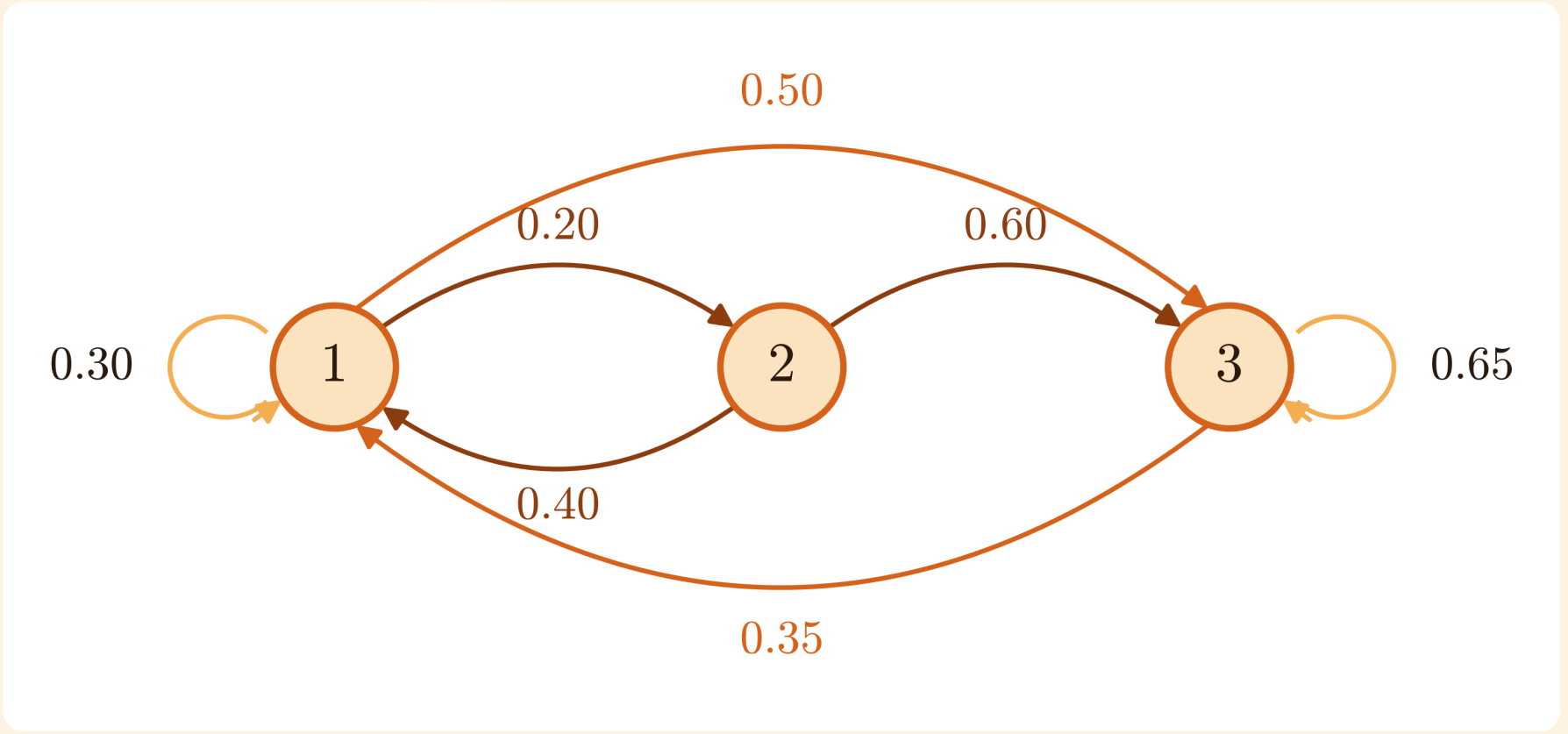
Transition matrix – example

- Consider a DTMC with the transition matrix

$$\mathbf{P} = \begin{bmatrix} 0.30 & 0.20 & 0.50 \\ 0.40 & 0.00 & 0.60 \\ 0.35 & 0.00 & 0.65 \end{bmatrix}$$

Transition diagram

- The **transition diagram** for the DTMC is shown below



Transition diagram

- Note that the rows of the transition probability matrix satisfy

$$\sum_{j=1}^{N} p_{ij} = 1$$

$$\mathbf{P} = \begin{bmatrix} 0.30 & 0.20 & 0.50 \\ 0.40 & 0.00 & 0.60 \\ 0.35 & 0.00 & 0.65 \end{bmatrix}$$

- e.g. row 1: $0.30 + 0.20 + 0.50 = 1$

DTMC example

- A company uses a machine that is either **up** or **down**.
- If it is up at the beginning of a day,
  - then it is up at the beginning of the next day with probability 0.95
  - or it fails with probability 0.05.

DTMC example

- On the other hand, if the machine is down at the beginning of a day,
  - it is down at the beginning of the next day with probability 0.10
  - or the repair is completed and the machine is up with probability 0.90.

DTMC example

- With state 0 = down and state 1 = up, the **transition probability matrix** of the DTMC is

$$\mathbf{P} = \begin{bmatrix} 0.10 & 0.90 \\ 0.05 & 0.95 \end{bmatrix}$$

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n-step transitions

Powers of the transition matrix



n-step transition

- The n-step transition probability matrix $\mathbf{P}^{(n)}$ has the conditional probabilities $P(X_n = j | X_0 = i)$
- This is given as

$$\mathbf{P}^{(n)} = \mathbf{P}^n$$

n-step transition – example

- Consider the company machine example.
- Compute the 2- and 10-step transition probability matrices for this DTMC

$$\mathbf{P} = \begin{bmatrix} 0.10 & 0.90 \\ 0.05 & 0.95 \end{bmatrix}$$

n-step transition – example

- These can be computed as

$$\mathbf{P}^2 = \begin{bmatrix} 0.0550 & 0.9450 \\ 0.0525 & 0.9475 \end{bmatrix}$$

$$\mathbf{P}^{10} = \begin{bmatrix} 0.05263158 & 0.94736842 \\ 0.05263158 & 0.94736842 \end{bmatrix}$$

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Limiting distribution

The steady state $\pi = \pi P$



Limiting distribution

- Examine

$$\mathbf{P}^{10} = \begin{bmatrix} 0.05263158 & 0.94736842 \\ 0.05263158 & 0.94736842 \end{bmatrix}$$

$$\mathbf{P}^{100} = \begin{bmatrix} 0.05263158 & 0.94736842 \\ 0.05263158 & 0.94736842 \end{bmatrix}$$

- Observe that $\mathbf{P}^{10} \approx \mathbf{P}^{100}$. This is explained by the **limiting distribution**.

Limiting distribution

- Does the **probability mass function (PMF)** of X_n approach a limit as n tends to infinity?

$$\lim_{n\to\infty} P\left(X_n = j\right) = ?$$

Limiting distribution

- If it does, we call it the **limiting** or **steady-state distribution** and denote it by

$$\bar{\boldsymbol{\pi}} = \begin{bmatrix} \pi_1 & \pi_2 & \dots & \pi_N \end{bmatrix}$$

$$\pi_j = \lim_{n \to \infty} P\left(X_n = j\right)$$

Limiting distribution

- If a limiting distribution exists, it is given as the solution to

$$\bar{\boldsymbol{\pi}} = \bar{\boldsymbol{\pi}}\mathbf{P}$$

- This can be seen as follows

$$P\left(X_{n+1}=j\right)=\sum_{i=1}^{N} P\left(X_n=i\right)p_{ij}$$

Limiting distribution

- Taking the limit $n \to \infty$ on both sides

$$\lim_{n\to\infty} P\left(X_{n+1}=j\right) = \lim_{n\to\infty} \sum_{i=1}^{N} P\left(X_n=i\right) p_{ij}$$

$$\Rightarrow \pi_j = \sum_{i=1}^{N} \pi_i p_{ij}$$

$$\Rightarrow \bar{\boldsymbol{\pi}} = \bar{\boldsymbol{\pi}}\mathbf{P}$$

Limiting distribution

- The limiting distribution can also be found from the powers of $\mathbf{P}$
- Consider the transition probability matrix

$$\mathbf{P} = \begin{bmatrix} 0.30 & 0.20 & 0.50 \\ 0.40 & 0.00 & 0.60 \\ 0.35 & 0.00 & 0.65 \end{bmatrix}$$

Limiting distribution

- The n-step transition probability matrices for small values of n are

$$n = 2 : \mathbf{P}^2 = \begin{bmatrix} 0.3450 & 0.0600 & 0.5950 \\ 0.3300 & 0.0800 & 0.5900 \\ 0.3325 & 0.0700 & 0.5975 \end{bmatrix}$$

$$n = 3 : \mathbf{P}^3 = \begin{bmatrix} 0.3358 & 0.0690 & 0.5952 \\ 0.3375 & 0.0660 & 0.5965 \\ 0.3369 & 0.0665 & 0.5966 \end{bmatrix}$$

Limiting distribution

- For higher values of n, the matrices are computed as

$$n = 5 : \mathbf{P}^5 = \begin{bmatrix} 0.3365 & 0.0673 & 0.5961 \\ 0.3366 & 0.0673 & 0.5962 \\ 0.3365 & 0.0673 & 0.5962 \end{bmatrix}$$

$$n \geq 6 : \mathbf{P}^n \approx \begin{bmatrix} 0.3365 & 0.0673 & 0.5962 \\ 0.3365 & 0.0673 & 0.5962 \\ 0.3365 & 0.0673 & 0.5962 \end{bmatrix}$$

Limiting distribution

- Note that all the rows of $\mathbf{P}^n$ are the same in the limit
  - This implies that the limiting distribution of X_n is the same regardless of the initial distribution

$$\bar{\boldsymbol{\pi}} = \begin{bmatrix} 0.3365 & 0.0673 & 0.5962 \end{bmatrix} = \begin{bmatrix} \frac{35}{104} & \frac{7}{104} & \frac{62}{104} \end{bmatrix}$$

Limiting distribution

- Let $\mathbf{P}^{(\infty)}$ be the n-step transition probability matrix as $n \to \infty$

$$\mathbf{P}^{(\infty)} = \lim_{n \to \infty} \mathbf{P}^{(n)}$$

- Thus, the limiting distribution is given by each row of $\mathbf{P}^{(\infty)}$.

Summary

- A **DTMC** satisfies the Markov property: the future depends only on the present state
- Its **transition matrix** $\mathbf{P}$ has entries p_{ij} and each row sums to 1
- The n-step transition matrix is $\mathbf{P}^{(n)} = \mathbf{P}^n$
- The **limiting distribution** solves $\bar{\boldsymbol{\pi}} = \bar{\boldsymbol{\pi}}\mathbf{P}$ and equals each row of $\mathbf{P}^{(\infty)}$



Thank You!

Questions?