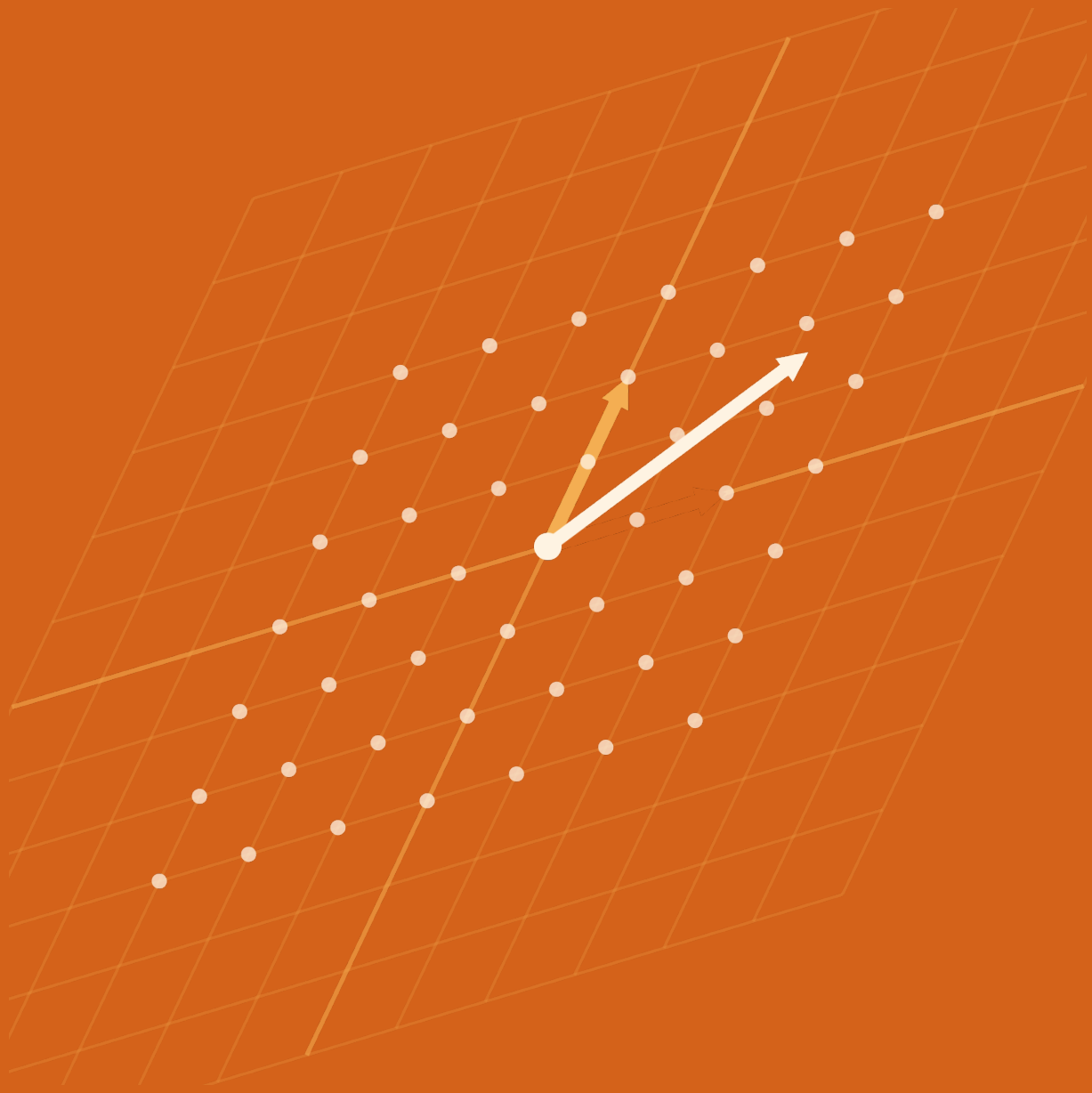
LINEAR ALGEBRA FOR AIML AND 6G

Linear Algebra and Probability



Lecture 24

Prof. Aditya K. Jagannatham
IIT Kanpur

Outline

1. **Gaussian random vectors**
   The Gaussian PDF in one and many dimensions
2. **i.i.d. random variables**
   Mean vector and covariance of i.i.d. samples
3. **Linear transformations**
   Mean and covariance of Ax + b

01

Gaussian random vectors

The Gaussian PDF in one and many dimensions



Gaussian density

- We now explore applications of linear algebra in probability and statistics
- Recall, the expression for the **Gaussian PDF** is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Gaussian density

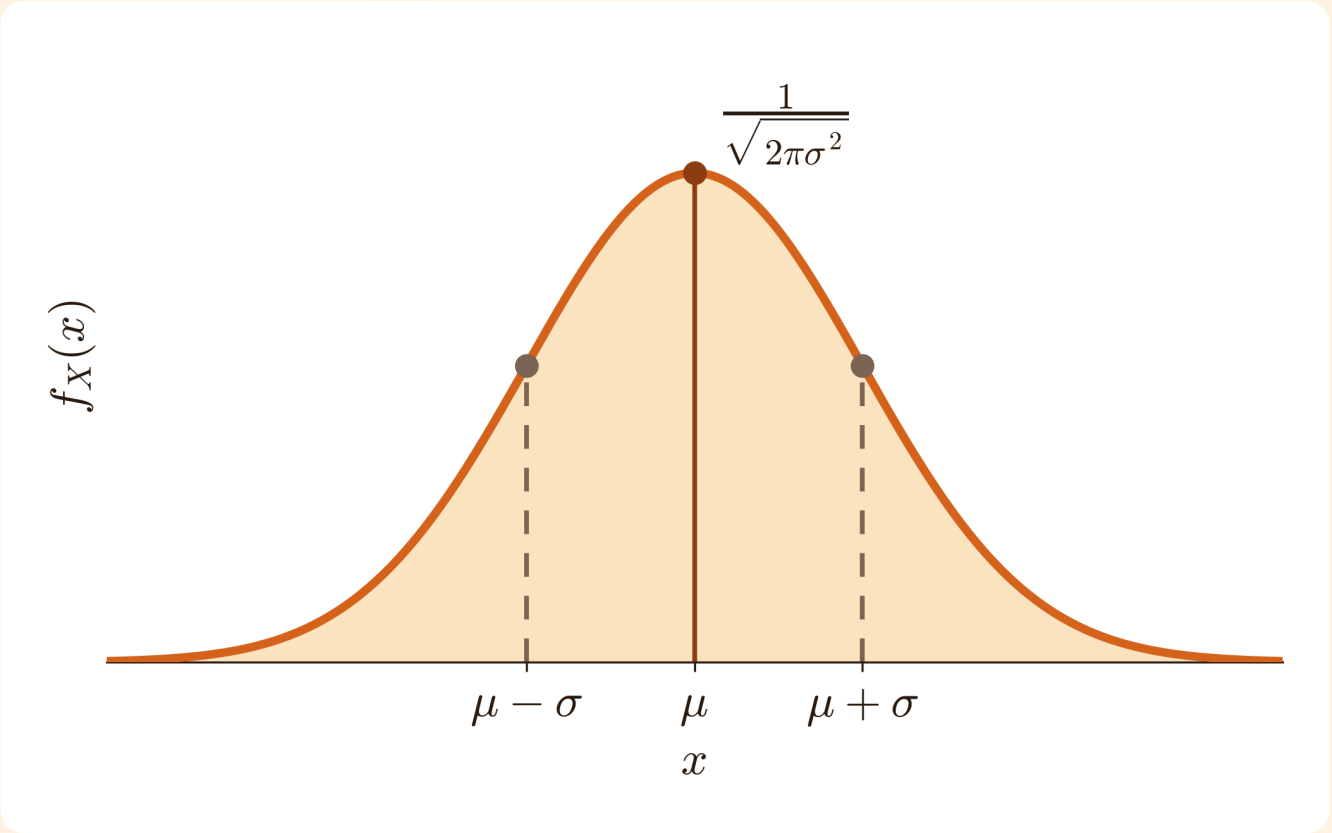
- The **mean** and **variance** of the Gaussian RV are

$$E\{X\} = \mu$$

$$E\left\{(X-\mu)^2\right\} = \sigma^2$$

Gaussian density

- The **Gaussian PDF** is shown below



Multivariate Gaussian

- Recall, the PDF of a **Gaussian random vector** is given as

$$f_{\bar{\mathbf{X}}}(\bar{\mathbf{x}}) = \frac{1}{\sqrt{(2\pi)^n |\mathbf{R}|}} e^{-\frac{1}{2}(\bar{\mathbf{x}}-\bar{\boldsymbol{\mu}})^T \mathbf{R}^{-1} (\bar{\mathbf{x}}-\bar{\boldsymbol{\mu}})}$$

Multivariate Gaussian

- The **mean** and **covariance matrix** are defined as

$$E\{\bar{\mathbf{x}}\} = \bar{\boldsymbol{\mu}}$$

$$E\left\{(\bar{\mathbf{x}} - \bar{\boldsymbol{\mu}})(\bar{\mathbf{x}} - \bar{\boldsymbol{\mu}})^T\right\} = \mathbf{R}$$

Multivariate Gaussian – example

- Determine the **multivariate Gaussian** PDF with mean and covariance matrix as below

$$\bar{\boldsymbol{\mu}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \mathbf{R} = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$$

Multivariate Gaussian – example

- We begin by evaluating $|\mathbf{R}|$ as follows

$$|\mathbf{R}| = \begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} = 8 - 1 = 7$$

- The quantity $\mathbf{R}^{-1}$ can be determined as

$$\mathbf{R}^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}$$

Multivariate Gaussian – example

- The **multivariate Gaussian** PDF is given as follows

$$\frac{1}{\sqrt{(2\pi)^n |\mathbf{R}|}} e^{-\frac{1}{2}(\bar{\mathbf{x}}-\bar{\boldsymbol{\mu}})^T \mathbf{R}^{-1} (\bar{\mathbf{x}}-\bar{\boldsymbol{\mu}})}$$

$$= \frac{1}{\sqrt{(2\pi)^2 \times 7}} \exp\left(-\frac{1}{14} \begin{bmatrix} x_1 - 2 & x_2 + 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 - 2 \\ x_2 + 1 \end{bmatrix} \right)$$

Multivariate Gaussian – example

- The quadratic form in the exponent simplifies as

$$\begin{bmatrix} x_1 - 2 & x_2 + 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 - 2 \\ x_2 + 1 \end{bmatrix}$$

$$= 2(x_1 - 2)^2 + 4(x_2 + 1)^2 - 2(x_1 - 2)(x_2 + 1)$$

$$= 2{x_1}^2 + 4{x_2}^2 - 10x_1 + 12x_2 - 2x_1x_2 + 16$$

Multivariate Gaussian – example

- The **multivariate Gaussian** PDF is now derived as

$$f\left(\bar{\mathbf{x}}\right) = \frac{1}{\sqrt{\left(2\pi\right)^2 \times 7}} e^{-\frac{1}{14}\left(2{x_1}^2+4{x_2}^2-10x_1+12x_2-2x_1x_2+16\right)}$$

02

i.i.d. random variables

Mean vector and covariance of i.i.d. samples



i.i.d. random variables

- Consider **i.i.d. random variables** $x_1, x_2, \ldots, x_n$ of mean 0 and variance σ^2
- Therefore

$$E\{x_i\} = 0$$

$$E\{x_i x_j\} = \begin{cases} \sigma^2 & i = j \\ 0 & i \neq j \end{cases}$$

i.i.d. random variables

- Let

$$\bar{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

i.i.d. random variables

- Then

$$E\{\bar{\mathbf{x}}\} = \begin{bmatrix} E\{x_1\} \\ E\{x_2\} \\ \vdots \\ E\{x_n\} \end{bmatrix} = \mathbf{0}$$

i.i.d. random variables

- The covariance matrix is

$$E\left\{\bar{\mathbf{x}}\bar{\mathbf{x}}^T\right\} = E\left\{\begin{bmatrix} x_1^2 & x_1x_2 & \dots & x_1x_n \\ x_2x_1 & x_2^2 & \dots & x_2x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_nx_1 & x_nx_2 & \dots & x_n^2 \end{bmatrix}\right\}$$

i.i.d. random variables

- The **covariance matrix** can be simplified as

$$E\left\{\bar{\mathbf{x}}\bar{\mathbf{x}}^T\right\} = \begin{bmatrix} \sigma^2 & 0 & \dots & 0 \\ 0 & \sigma^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma^2 \end{bmatrix}$$

$$= \sigma^2 \mathbf{I}$$

03

Linear transformations

Mean and covariance of Ax + b



Linear transformation

- Consider $\bar{\mathbf{y}} = \mathbf{A}\bar{\mathbf{x}} + \bar{\mathbf{b}}$, where $\bar{\mathbf{x}}$ has mean $\bar{\boldsymbol{\mu}}$ and covariance $\mathbf{R}$
- The **mean** $\bar{\boldsymbol{\mu}}_y$ of $\bar{\mathbf{y}}$ is obtained as follows

$$
\begin{aligned}
E\{\bar{\mathbf{y}}\} &= E\left\{\mathbf{A}\bar{\mathbf{x}} + \bar{\mathbf{b}}\right\} \\
&= \mathbf{A}E\{\bar{\mathbf{x}}\} + \bar{\mathbf{b}} \\
&= \mathbf{A}\bar{\boldsymbol{\mu}} + \bar{\mathbf{b}} = \bar{\boldsymbol{\mu}}_y
\end{aligned}
$$

Linear transformation

- Note that

$$\bar{\mathbf{y}} - \bar{\boldsymbol{\mu}}_y = \mathbf{A}\bar{\mathbf{x}} + \bar{\mathbf{b}} - \mathbf{A}\bar{\boldsymbol{\mu}} - \bar{\mathbf{b}}$$
$$= \mathbf{A}\left(\bar{\mathbf{x}} - \bar{\boldsymbol{\mu}}\right)$$

Linear transformation

- The **covariance matrix** is

$$E\left\{\left(\bar{\mathbf{y}}-\bar{\boldsymbol{\mu}}_y\right)\left(\bar{\mathbf{y}}-\bar{\boldsymbol{\mu}}_y\right)^T\right\}$$

$$=E\left\{\mathbf{A}\left(\bar{\mathbf{x}}-\bar{\boldsymbol{\mu}}\right)\left(\mathbf{A}\left(\bar{\mathbf{x}}-\bar{\boldsymbol{\mu}}\right)\right)^T\right\}$$

$$=\mathbf{A}E\left\{\left(\bar{\mathbf{x}}-\bar{\boldsymbol{\mu}}\right)\left(\bar{\mathbf{x}}-\bar{\boldsymbol{\mu}}\right)^T\right\}\mathbf{A}^T$$

$$=\mathbf{A}\mathbf{R}\mathbf{A}^T$$

Linear transformation

- If $\bar{\mathbf{x}}$ is Gaussian, then any **linear transformation** is also Gaussian

$$\bar{\mathbf{y}} = \mathbf{A}\bar{\mathbf{x}} + \bar{\mathbf{b}}$$

Linear transformation – example

- Consider a 2-dimensional random vector $\bar{\mathbf{x}}$ with a **multivariate Gaussian distribution** of mean $\bar{\boldsymbol{\mu}}$ and covariance $\mathbf{R}$

$$\bar{\boldsymbol{\mu}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \mathbf{R} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

Linear transformation – example

- Consider the vector $\bar{\mathbf{y}}$ defined as

$$\bar{\mathbf{y}} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \bar{\mathbf{x}} + \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

Linear transformation – example

- The **mean** $\bar{\boldsymbol{\mu}}_y$ is given as

$$\bar{\boldsymbol{\mu}}_y = \mathbf{A}\bar{\boldsymbol{\mu}} + \bar{\mathbf{b}}$$

$$= \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

Linear transformation – example

- The **covariance matrix** $\mathbf{R}_y$ is given as

$$\mathbf{R}_y = \mathbf{A}\mathbf{R}\mathbf{A}^T = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 3 & 12 \end{bmatrix}$$

Special case

- We explore the special case of a linear transformation of **i.i.d. random variables**
- Consider now **i.i.d. random variables** x_i with mean 0 and variance σ^2
- Let

$$y = \bar{\mathbf{a}}^T \bar{\mathbf{x}}$$

Special case

- Then

$$E\{y\} = \bar{\mathbf{a}}^T E\{\bar{\mathbf{x}}\} = 0$$

$$E\{y^2\} = E\left\{\left(\bar{\mathbf{a}}^T\bar{\mathbf{x}}\right)^2\right\}$$

$$= E\left\{\left(\bar{\mathbf{a}}^T\bar{\mathbf{x}}\right)\left(\bar{\mathbf{a}}^T\bar{\mathbf{x}}\right)^T\right\}$$

$$= E\left\{\bar{\mathbf{a}}^T\bar{\mathbf{x}}\bar{\mathbf{x}}^T\bar{\mathbf{a}}\right\} = \bar{\mathbf{a}}^T E\left\{\bar{\mathbf{x}}\bar{\mathbf{x}}^T\right\}\bar{\mathbf{a}}$$

Special case

- This can be simplified as

$$E\left\{y^2\right\} = \bar{\mathbf{a}}^T E\left\{\bar{\mathbf{x}}\bar{\mathbf{x}}^T\right\}\bar{\mathbf{a}}$$
$$= \bar{\mathbf{a}}^T\sigma^2\mathbf{I}\bar{\mathbf{a}} = \sigma^2\bar{\mathbf{a}}^T\bar{\mathbf{a}}$$
$$= \sigma^2\|\bar{\mathbf{a}}\|^2$$

Summary

- A Gaussian random vector has PDF $(1/(\sqrt{((2\pi)^n|\mathbf{R}|)}))e^{-(1/2)(\bar{\mathbf{x}}-\bar{\boldsymbol{\mu}})^T\mathbf{R}^{-1}(\bar{\mathbf{x}}-\bar{\boldsymbol{\mu}})}$
- Its mean is $\bar{\boldsymbol{\mu}}$ and its covariance matrix is $\mathbf{R} = E\{(\bar{\mathbf{x}} - \bar{\boldsymbol{\mu}})(\bar{\mathbf{x}} - \bar{\boldsymbol{\mu}})^T\}$
- i.i.d. zero-mean samples of variance σ^2 have covariance $\sigma^2\mathbf{I}$
- $\bar{\mathbf{y}} = \mathbf{A}\bar{\mathbf{x}} + \bar{\mathbf{b}}$ has mean $\mathbf{A}\bar{\boldsymbol{\mu}} + \bar{\mathbf{b}}$ and covariance $\mathbf{A}\mathbf{R}\mathbf{A}^T$; it stays Gaussian
- For i.i.d. $\bar{\mathbf{x}}$, $y = \bar{\mathbf{a}}^T\bar{\mathbf{x}}$ has variance $\sigma^2\|\bar{\mathbf{a}}\|^2$



Thank You!

Questions?