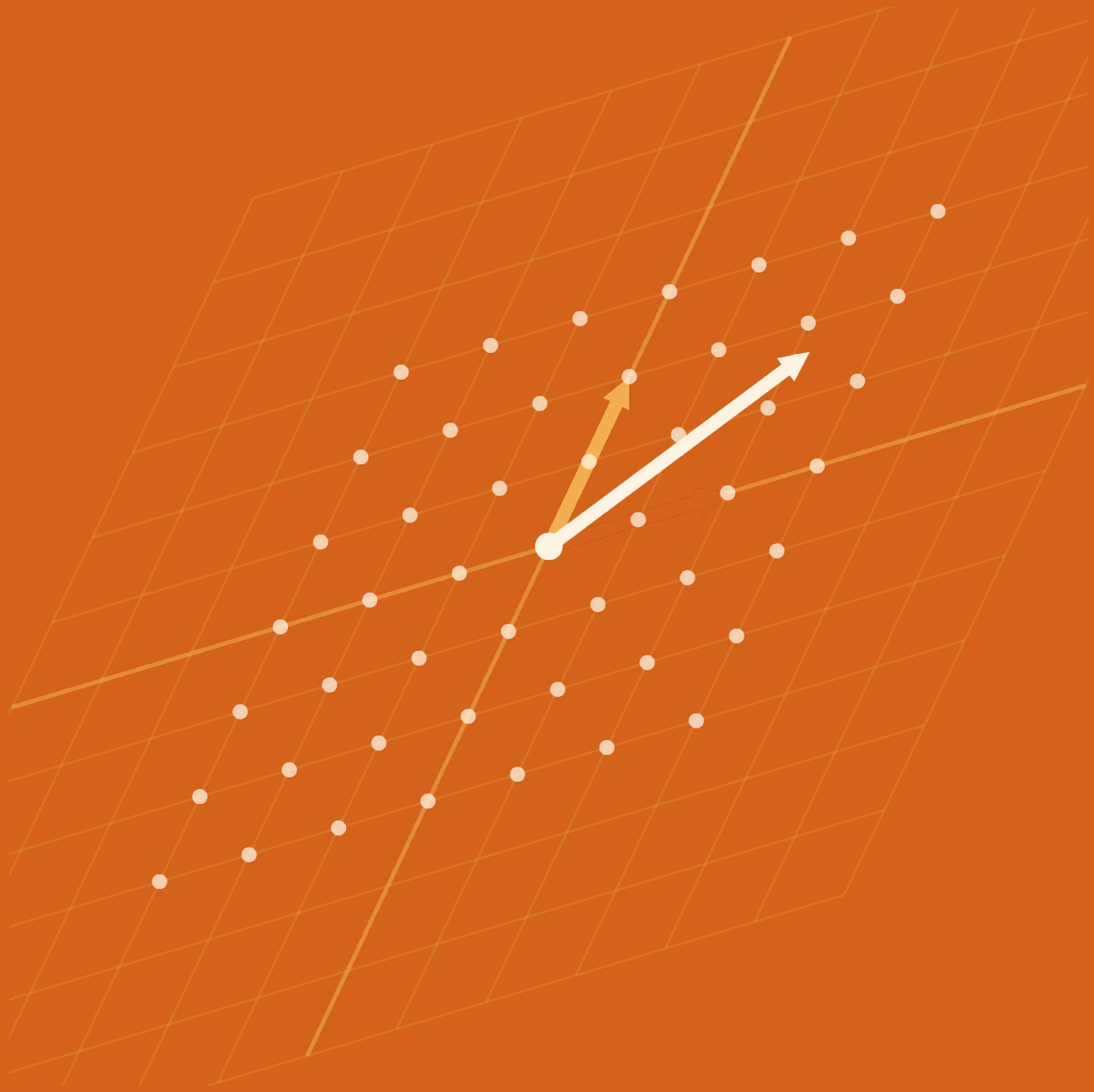
LINEAR ALGEBRA FOR AIML AND 6G

Woodbury Matrix Identity



Lecture 23

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Outline

1. **Woodbury matrix identity**
   Statement of the identity and two helper identities
2. **Proof of the WMI**
   Deriving the identity step by step
3. **Application and example**
   Inverting a rank-one update of the identity

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Woodbury matrix identity

Statement of the identity and two helper identities



Woodbury matrix identity

- The **Woodbury matrix identity (WMI)** is stated as

$$(\mathbf{A}+\mathbf{UCV})^{-1}$$

$$=\mathbf{A}^{-1}-\mathbf{A}^{-1}\mathbf{U}\left(\mathbf{C}^{-1}+\mathbf{VA}^{-1}\mathbf{U}\right)^{-1}\mathbf{VA}^{-1}$$

Identity 1

- We start by proving the following identity

$$(\mathbf{I}+\mathbf{P})^{-1} = \mathbf{I} - (\mathbf{I}+\mathbf{P})^{-1}\mathbf{P}$$

Identity 1

- This can be seen as follows

$$
\begin{aligned}
&(\mathbf{I}+\mathbf{P})^{-1} \\
&= (\mathbf{I}+\mathbf{P})^{-1}\mathbf{I} \\
&= (\mathbf{I}+\mathbf{P})^{-1}(\mathbf{I}+\mathbf{P}-\mathbf{P}) \\
&= \mathbf{I}-(\mathbf{I}+\mathbf{P})^{-1}\mathbf{P}
\end{aligned}
$$

Identity 2

- Consider now the identity

$$(\mathbf{I}+\mathbf{PQ})^{-1}\mathbf{P}=\mathbf{P}(\mathbf{I}+\mathbf{QP})^{-1}$$

Identity 2

- This can be proved as follows

$$\mathbf{P} + \mathbf{PQP} = \mathbf{P} + \mathbf{PQP}$$
$$\Rightarrow \mathbf{P}\left(\mathbf{I} + \mathbf{QP}\right) = \left(\mathbf{I} + \mathbf{PQ}\right)\mathbf{P}$$
$$\Rightarrow \left(\mathbf{I} + \mathbf{PQ}\right)^{-1}\mathbf{P} = \mathbf{P}\left(\mathbf{I} + \mathbf{QP}\right)^{-1}$$

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Proof of the WMI

Deriving the identity step by step



Proof of the WMI

- We now start with the proof of the **Woodbury matrix identity**
- Note that

$$(\mathbf{A}+\mathbf{UCV})^{-1} = \left(\mathbf{A}\left(\mathbf{I}+\mathbf{A}^{-1}\mathbf{UCV}\right)\right)^{-1}$$

$$= \left(\mathbf{I}+\mathbf{A}^{-1}\mathbf{UCV}\right)^{-1}\mathbf{A}^{-1}$$

Proof of the WMI

- Use identity 1 with $\mathbf{P} = \mathbf{A}^{-1}\mathbf{UCV}$,

$$
\begin{aligned}
&(\mathbf{A} + \mathbf{UCV})^{-1} \\
&= \left(\mathbf{I} + \mathbf{A}^{-1}\mathbf{UCV}\right)^{-1}\mathbf{A}^{-1} \\
&= \left(\mathbf{I} - \left(\mathbf{I} + \mathbf{A}^{-1}\mathbf{UCV}\right)^{-1}\mathbf{A}^{-1}\mathbf{UCV}\right)\mathbf{A}^{-1} \\
&= \mathbf{A}^{-1} - \left(\mathbf{I} + \mathbf{A}^{-1}\mathbf{UCV}\right)^{-1}\mathbf{A}^{-1}\mathbf{UCVA}^{-1}
\end{aligned}
$$

Proof of the WMI

- Use identity 2 with $\mathbf{P} = \mathbf{A}^{-1}\mathbf{U}$ and $\mathbf{Q} = \mathbf{CV}$,

$$
\begin{aligned}
&(\mathbf{A} + \mathbf{UCV})^{-1} \\
&= \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}\left(\mathbf{I} + \mathbf{CVA}^{-1}\mathbf{U}\right)^{-1}\mathbf{CVA}^{-1} \\
&= \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}\left(\mathbf{C}\left(\mathbf{C}^{-1} + \mathbf{VA}^{-1}\mathbf{U}\right)\right)^{-1}\mathbf{CVA}^{-1} \\
&= \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}\left(\mathbf{C}^{-1} + \mathbf{VA}^{-1}\mathbf{U}\right)^{-1}\mathbf{VA}^{-1}
\end{aligned}
$$

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Application and example

Inverting a rank-one update of the identity



WMI – application

- Using the WMI, we simplify

$$\left(\mathbf{I} + \bar{\mathbf{x}}\bar{\mathbf{x}}^T\right)^{-1}$$

WMI – application

- This can be simplified as follows.

$$\mathbf{A} = \mathbf{I}, \mathbf{U} = \bar{\mathbf{x}}, \mathbf{V} = \bar{\mathbf{x}}^T, \mathbf{C} = 1$$

$$\left(\mathbf{I} + \bar{\mathbf{x}}\bar{\mathbf{x}}^T\right)^{-1}$$

$$= \mathbf{I} - \mathbf{I}\bar{\mathbf{x}}\left(1^{-1} + \bar{\mathbf{x}}^T\mathbf{I}^{-1}\bar{\mathbf{x}}\right)^{-1}\bar{\mathbf{x}}^T\mathbf{I}$$

$$= \mathbf{I} - \frac{\bar{\mathbf{x}}\bar{\mathbf{x}}^T}{1 + \bar{\mathbf{x}}^T\bar{\mathbf{x}}} = \mathbf{I} - \frac{\bar{\mathbf{x}}\bar{\mathbf{x}}^T}{1 + \|\bar{\mathbf{x}}\|^2}$$

WMI – example

- Consider the matrix below

$$\begin{bmatrix} 2 & -1 & 1 & 2 \\ -1 & 2 & -1 & -2 \\ 1 & -1 & 2 & 2 \\ 2 & -2 & 2 & 5 \end{bmatrix}$$

WMI – example

- The matrix can be expressed as

$$\begin{bmatrix} 2 & -1 & 1 & 2 \\ -1 & 2 & -1 & -2 \\ 1 & -1 & 2 & 2 \\ 2 & -2 & 2 & 5 \end{bmatrix} = \mathbf{I} + \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & 2 \end{bmatrix} = \mathbf{I} + \bar{\mathbf{x}}\bar{\mathbf{x}}^T$$

WMI – example

- Therefore, using the property

$$\left(\mathbf{I}+\bar{\mathbf{x}}\bar{\mathbf{x}}^{T}\right)^{-1}=\mathbf{I}-\frac{\bar{\mathbf{x}}\bar{\mathbf{x}}^{T}}{1+\left\|\bar{\mathbf{x}}\right\|^{2}}$$

- with $\left\|\bar{\mathbf{x}}\right\|^{2}=1+1+1+4=7$

WMI – example

- The inverse can be evaluated as

$$\begin{bmatrix} 2 & -1 & 1 & 2 \\ -1 & 2 & -1 & -2 \\ 1 & -1 & 2 & 2 \\ 2 & -2 & 2 & 5 \end{bmatrix}^{-1} = \mathbf{I} - \frac{1}{8} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & 2 \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} 7 & 1 & -1 & -2 \\ 1 & 7 & 1 & 2 \\ -1 & 1 & 7 & -2 \\ -2 & 2 & -2 & 4 \end{bmatrix}$$

Summary

- **WMI**: $(\mathbf{A}+\mathbf{UCV})^{-1}=\mathbf{A}^{-1}-\mathbf{A}^{-1}\mathbf{U}(\mathbf{C}^{-1}+\mathbf{VA}^{-1}\mathbf{U})^{-1}\mathbf{VA}^{-1}$
- Identity 1: $(\mathbf{I}+\mathbf{P})^{-1}=\mathbf{I}-(\mathbf{I}+\mathbf{P})^{-1}\mathbf{P}$
- Identity 2: $(\mathbf{I}+\mathbf{PQ})^{-1}\mathbf{P}=\mathbf{P}(\mathbf{I}+\mathbf{QP})^{-1}$
- Rank-one update: $(\mathbf{I}+\bar{\mathbf{x}}\bar{\mathbf{x}}^T)^{-1}=\mathbf{I}-(\bar{\mathbf{x}}\bar{\mathbf{x}}^T)/(1+\|\bar{\mathbf{x}}\|^2)$



Thank You!

Questions?