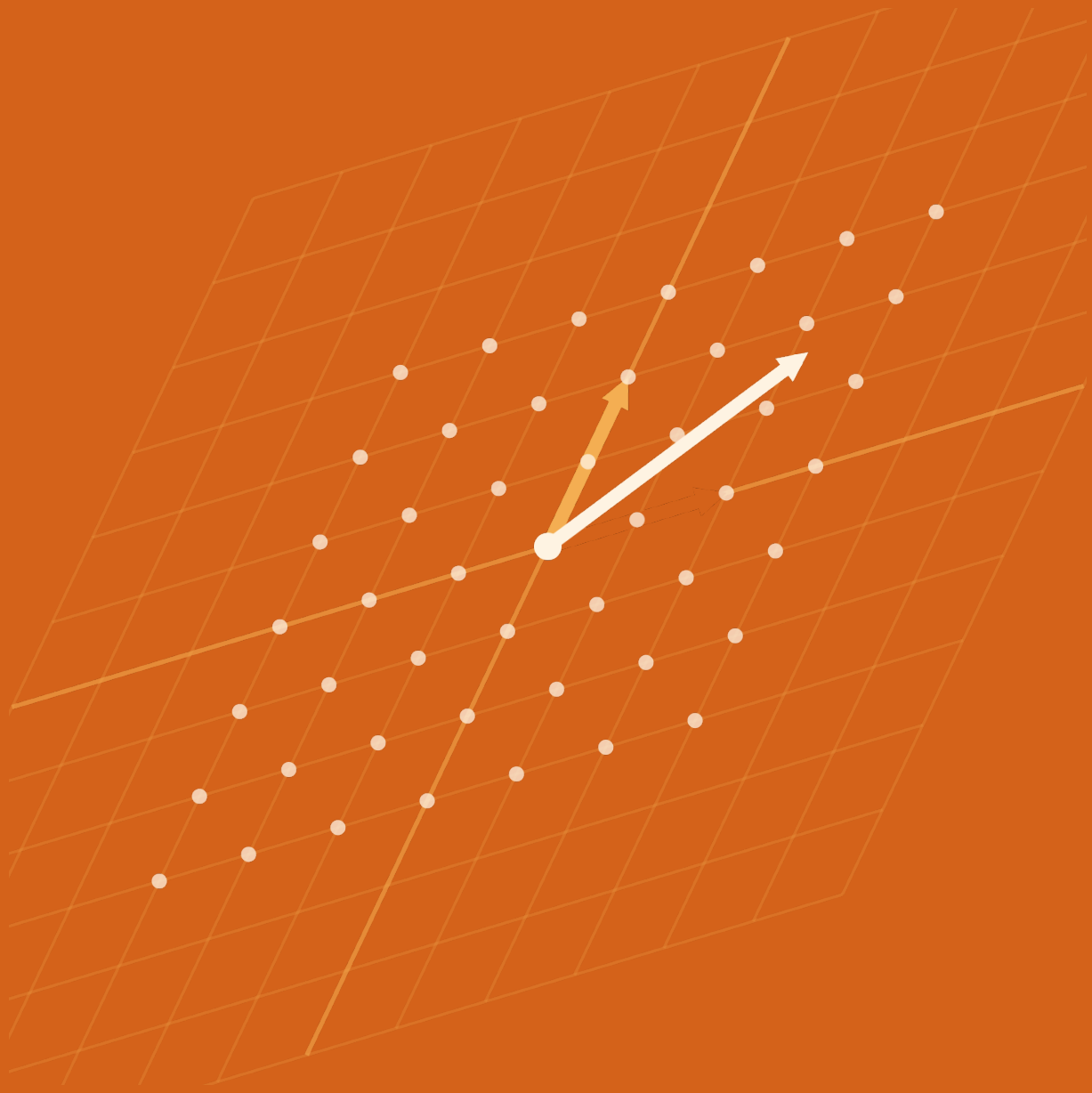
LINEAR ALGEBRA FOR AIML AND 6G

Properties of the Eigenvalue Decomposition

Lecture 17

Prof. Aditya K. Jagannatham
IIT Kanpur



Outline

1. **Trace and determinant properties**
   Trace and determinant from the eigenvalues
2. **Symmetric matrices**
   Real eigenvalues and orthonormal eigenvectors
3. **Positive definite and semidefinite matrices**
   Definiteness and the signs of the eigenvalues

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Trace and determinant properties

Trace and determinant from the eigenvalues



Trace property

- The **trace of the matrix** is given as

$$\begin{aligned}\text{trace}\left(\mathbf{A}\right) &= \text{trace}\left(\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}\right)\\ &= \text{trace}\left(\mathbf{\Lambda}\mathbf{V}^{-1}\mathbf{V}\right)\\ &= \text{trace}\left(\mathbf{\Lambda}\right) = \sum_{i=1}^{n}\lambda_i\end{aligned}$$

  - Thus **trace** of a matrix equals the **sum of its eigenvalues**

Trace – example

- Consider the 3×3 matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 \\ -1 & 0 & -1 \\ 2 & 4 & 4 \end{bmatrix}$$

$$\text{trace}\,(\mathbf{A}) = 3 + 0 + 4 = 7$$

Trace – example

- The eigenvalues of $\mathbf{A}$ are $2,2,3$ (see Lecture 16), so

$$\sum_{i=1}^{3} \lambda_i = 2 + 2 + 3 = 7$$

Determinant property

- The determinant of the matrix is given as

$$\det(\mathbf{A}) = \det\left(\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}\right)$$
$$= \det(\mathbf{V})\det(\mathbf{\Lambda})\det\left(\mathbf{V}^{-1}\right)$$
$$= \det\mathbf{\Lambda} = \prod_{i=1}^{n}\lambda_i$$

  - Thus **determinant** of a matrix equals the **product of its eigenvalues**

Determinant – example

- Consider the 3×3 matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 \\ -1 & 0 & -1 \\ 2 & 4 & 4 \end{bmatrix}$$

Determinant – example

- The determinant is calculated using **Laplace formula** as

$$\begin{aligned}\det(\mathbf{A}) &= 3(0+4) - 2(-4+2) + 1(-4-0) \\ &= 12 + 4 - 4 = 12\end{aligned}$$

- This equals the product of eigenvalues evaluated as

$$= \prod_{i=1}^{3} \lambda_i = 2 \times 2 \times 3 = 12$$

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Symmetric matrices

Real eigenvalues and orthonormal eigenvectors



Symmetric matrix

- An important property of eigenvalues of a symmetric matrix is below
- Consider a **symmetric (Hermitian) matrix** $\mathbf{A}$

$$\mathbf{A} = \mathbf{A}^H$$

- Its eigenvalues are real i.e. $\lambda \in \mathbb{R}$

Symmetric matrix

- This can be seen as follows
- Let λ be the **eigenvalue** with $\bar{\mathbf{v}}$ as the eigenvector
- Then

$$\mathbf{A}\bar{\mathbf{v}} = \lambda\bar{\mathbf{v}}$$

$$\bar{\mathbf{v}}^H\mathbf{A}\bar{\mathbf{v}} = \bar{\mathbf{v}}^H\lambda\bar{\mathbf{v}} = \lambda\left\|\bar{\mathbf{v}}\right\|^2$$

$$\Rightarrow \lambda = \frac{\bar{\mathbf{v}}^H\mathbf{A}\bar{\mathbf{v}}}{\left\|\bar{\mathbf{v}}\right\|^2}$$

Symmetric matrix

- Further,

$$\left(\bar{\mathbf{v}}^H \mathbf{A} \bar{\mathbf{v}}\right)^* = \left(\bar{\mathbf{v}}^H \mathbf{A} \bar{\mathbf{v}}\right)^H = \bar{\mathbf{v}}^H \mathbf{A}^H \bar{\mathbf{v}}$$

$$= \bar{\mathbf{v}}^H \mathbf{A} \bar{\mathbf{v}}$$

$$\Rightarrow \bar{\mathbf{v}}^H \mathbf{A} \bar{\mathbf{v}} \in \mathbb{R}$$

Symmetric matrix

- Therefore, since $\bar{\mathbf{v}}^H \mathbf{A} \bar{\mathbf{v}}$ and $\|\bar{\mathbf{v}}\|^2$ are real

$$\lambda = \frac{\bar{\mathbf{v}}^H \mathbf{A} \bar{\mathbf{v}}}{\|\bar{\mathbf{v}}\|^2} \in \mathbb{R}$$

Eigenvectors

- Property: Consider a **symmetric matrix** $\mathbf{A}$
- Its eigenvectors corresponding to distinct **eigenvalues** are **orthogonal**

Eigenvectors

- This can be seen as below

$$\mathbf{A}\bar{\mathbf{u}} = \lambda\bar{\mathbf{u}}$$
$$\mathbf{A}\bar{\mathbf{v}} = \mu\bar{\mathbf{v}}$$
$$\bar{\mathbf{v}}^H\mathbf{A}\bar{\mathbf{u}} = \bar{\mathbf{v}}^H\lambda\bar{\mathbf{u}}$$
$$\bar{\mathbf{u}}^H\mathbf{A}\bar{\mathbf{v}} = \bar{\mathbf{u}}^H\mu\bar{\mathbf{v}}$$
$$\left(\bar{\mathbf{u}}^H\mathbf{A}\bar{\mathbf{v}}\right)^H = \bar{\mathbf{v}}^H\mathbf{A}^H\bar{\mathbf{u}} = \bar{\mathbf{v}}^H\mathbf{A}\bar{\mathbf{u}} = \bar{\mathbf{v}}^H\lambda\bar{\mathbf{u}}$$

Eigenvectors

- Furthermore, since $\mu \in \mathbb{R}$, we have $\mu = \mu^*$.
- Therefore, it follows that

$$\left(\bar{\mathbf{u}}^H \mathbf{A} \bar{\mathbf{v}}\right)^H = \left(\bar{\mathbf{u}}^H \mu \bar{\mathbf{v}}\right)^H$$
$$= \bar{\mathbf{v}}^H \mu^* \bar{\mathbf{u}} = \bar{\mathbf{v}}^H \mu \bar{\mathbf{u}}$$

Eigenvectors

- Thus, it can be deduced that

$$\bar{\mathbf{v}}^H \mu \bar{\mathbf{u}} = \bar{\mathbf{v}}^H \lambda \bar{\mathbf{u}}$$
$$\Rightarrow (\lambda - \mu)\, \bar{\mathbf{v}}^H \bar{\mathbf{u}} = 0$$

- Since $\lambda \neq \mu$, it follows that

$$\bar{\mathbf{v}}^H \bar{\mathbf{u}} = 0$$

Eigenvalue decomposition

- For a symmetric $n \times n$ matrix, there is an **orthonormal** basis of $\mathbb{R}^n$ consisting of n eigenvectors of $\mathbf{A}$
- This means, the eigenvectors satisfy

$$\bar{\mathbf{v}}_i^H \bar{\mathbf{v}}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Eigenvalue decomposition

$$\mathbf{V} = \begin{bmatrix} | & | & & | \\ \bar{\mathbf{v}}_1 & \bar{\mathbf{v}}_2 & \dots & \bar{\mathbf{v}}_n \\ | & | & & | \end{bmatrix}$$

- Therefore, the matrix $\mathbf{V}$ of **eigenvectors** satisfies

$$\mathbf{V}\mathbf{V}^H = \mathbf{V}^H\mathbf{V} = \mathbf{I}$$

Eigenvalue decomposition

- Therefore, the **eigenvalue decomposition** is given as

$$\begin{aligned}\mathbf{A} &= \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1} \\ &= \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{H}\end{aligned}$$

Eigenvalue decomposition

- For a **real symmetric matrix**

$$\mathbf{A} = \mathbf{A}^T$$

$$\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^T$$

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Positive definite and semidefinite matrices

Definiteness and the signs of the eigenvalues



Eigenvalues PSD matrix

- Consider a symmetric matrix $\mathbf{A} = \mathbf{A}^T$
- Recall, it is **positive semidefinite** if for every $\bar{\mathbf{x}}$

$$\bar{\mathbf{x}}^T \mathbf{A} \bar{\mathbf{x}} \geq 0$$

Eigenvalues PSD matrix

- Eigenvalues of a **positive semidefinite matrix** are non-negative
- This can be seen as follows

$$\mathbf{A}\bar{\mathbf{x}} = \lambda\bar{\mathbf{x}}$$

$$\bar{\mathbf{x}}^T\mathbf{A}\bar{\mathbf{x}} = \bar{\mathbf{x}}^T\lambda\bar{\mathbf{x}} = \lambda\|\bar{\mathbf{x}}\|^2$$

Eigenvalues PSD matrix

- Note that $\bar{\mathbf{x}}^T\mathbf{A}\bar{\mathbf{x}} \geq 0$ and $\|\bar{\mathbf{x}}\|^2 > 0$.
- Therefore,

$$\lambda = \frac{\bar{\mathbf{x}}^T\mathbf{A}\bar{\mathbf{x}}}{\|\bar{\mathbf{x}}\|^2} \geq 0$$

Eigenvalues PD matrix

- Similarly, since $\bar{\mathbf{x}}^T\mathbf{A}\bar{\mathbf{x}} > 0$ for a positive definite matrix $\mathbf{A}$,
  - its eigenvalues satisfy

$$\lambda = \frac{\bar{\mathbf{x}}^T\mathbf{A}\bar{\mathbf{x}}}{\|\bar{\mathbf{x}}\|^2} > 0$$

Positive eigenvalues

- Conversely, if a **symmetric matrix** $\mathbf{A}$ has positive eigenvalues λ_i, then it is **positive definite**
- Consider **orthonormal eigenvectors** $\bar{\mathbf{v}}_i$

$$\bar{\mathbf{v}}_i^T \bar{\mathbf{v}}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Positive eigenvalues

- Any vector $\bar{\mathbf{x}}$ can be expressed as

$$\bar{\mathbf{x}} = c_1\bar{\mathbf{v}}_1 + c_2\bar{\mathbf{v}}_2 + \ldots + c_n\bar{\mathbf{v}}_n$$

- Therefore, it follows that

$$\begin{aligned}\bar{\mathbf{x}}^T\mathbf{A}\bar{\mathbf{x}} &= \left(c_1\bar{\mathbf{v}}_1 + c_2\bar{\mathbf{v}}_2 + \ldots + c_n\bar{\mathbf{v}}_n\right)^T \\ &\times\mathbf{A}\left(c_1\bar{\mathbf{v}}_1 + c_2\bar{\mathbf{v}}_2 + \ldots + c_n\bar{\mathbf{v}}_n\right) \\ &= \left(c_1\bar{\mathbf{v}}_1 + c_2\bar{\mathbf{v}}_2 + \ldots + c_n\bar{\mathbf{v}}_n\right)^T \\ &\times\left(c_1\lambda_1\bar{\mathbf{v}}_1 + c_2\lambda_2\bar{\mathbf{v}}_2 + \ldots + c_n\lambda_n\bar{\mathbf{v}}_n\right)\end{aligned}$$

Positive eigenvalues

- This can be simplified as

$$\bar{\mathbf{x}}^T \mathbf{A} \bar{\mathbf{x}} = \lambda_1 c_1^2 + \lambda_2 c_2^2 + \ldots + \lambda_n c_n^2$$

- For $\bar{\mathbf{x}} \neq \mathbf{0}$, not all c_i are zero, so the RHS is > 0 since $\boldsymbol{\lambda}_i > 0$.
- Therefore, for any nonzero vector $\bar{\mathbf{x}}$

$$\bar{\mathbf{x}}^T \mathbf{A} \bar{\mathbf{x}} > 0$$

  - which implies $\mathbf{A}$ is a **PD matrix**

Summary

- The **trace** of a diagonalizable matrix equals the sum of its eigenvalues
- The **determinant** equals the product of the eigenvalues
- A symmetric (Hermitian) matrix $\mathbf{A} = \mathbf{A}^H$ has real eigenvalues
- Its eigenvectors for distinct eigenvalues are orthogonal, so $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^H$ with $\mathbf{V}\mathbf{V}^H = \mathbf{I}$
- A symmetric matrix is PSD (PD) if and only if all its eigenvalues are ≥ 0 (> 0)



Thank You!

Questions?