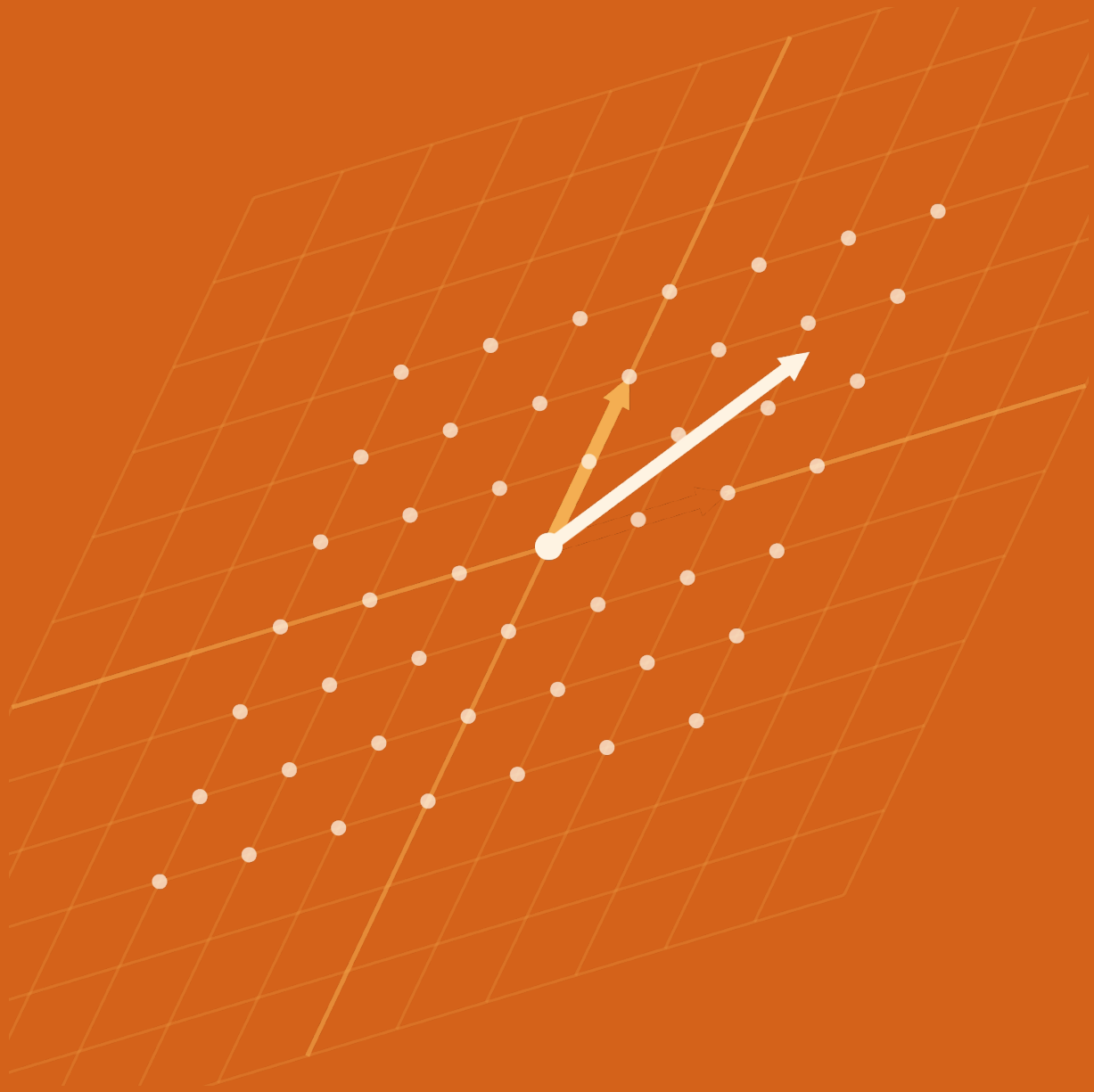
LINEAR ALGEBRA FOR AIML AND 6G

Eigenvalues, Eigenvectors and EVD



Lecture 16

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Outline

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   Definition and the characteristic equation
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Eigenvalues and eigenvectors

Definition and the characteristic equation



Eigenvalues

- Let $\mathbf{A}$ be an $n \times n$ matrix
- A scalar λ is called an **eigenvalue** of $\mathbf{A}$ if there is a non-zero vector $\bar{\mathbf{v}} \neq \mathbf{0}$ called an **eigenvector** such that

$$\mathbf{A}\bar{\mathbf{v}} = \lambda\bar{\mathbf{v}}$$

Eigenvalues

- We can rewrite this as

$$\mathbf{A}\bar{\mathbf{v}} = \lambda\mathbf{I}\bar{\mathbf{v}}$$
$$\Rightarrow (\mathbf{A} - \lambda\mathbf{I})\,\bar{\mathbf{v}} = \mathbf{0}$$

Eigenvalues

- Therefore, λ is an **eigenvalue** of $\mathbf{A}$ if and only if $(\mathbf{A} - \lambda\mathbf{I})$ is **singular**
- This implies a scalar λ is an **eigenvalue** of the matrix $\mathbf{A}$
  - if and only if λ is a solution of the **characteristic equation**

$$|\mathbf{A} - \lambda\mathbf{I}| = 0$$

Eigenvectors

- The corresponding **eigenvectors** are solutions to the equations

$$(\mathbf{A} - \lambda \mathbf{I})\,\bar{\mathbf{v}} = \mathbf{0}$$

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Worked examples



A 2 × 2 matrix, and a 3 × 3 matrix with a repeated eigenvalue

Eigenvalues – example 1

- Determine the **eigenvalues** and **eigenvectors** of the matrix below

$$\mathbf{A} = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$$

Eigenvalues – example 1

- The eigenvalues can be found as below

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (5-\lambda)^2 - 4 = 0$$

$$\Rightarrow \lambda^2 - 10\lambda + 21 = 0$$

$$\Rightarrow (\lambda - 7)(\lambda - 3) = 0$$

$$\Rightarrow \lambda = 3, 7$$

Eigenvalues – example 1

- **Eigenvectors** can be found by solving the homogeneous system

$$(\mathbf{A} - \lambda \mathbf{I})\,\bar{\mathbf{v}} = \mathbf{0}$$

$$(\mathbf{A} - 7\mathbf{I})\,\bar{\mathbf{v}} = \mathbf{0}$$

$$\Rightarrow \left(\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \right) \bar{\mathbf{v}} = \mathbf{0}$$

$$\Rightarrow \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \bar{\mathbf{v}} = \mathbf{0}$$

Eigenvalues – example 1

- Simplifying further

$$\Rightarrow \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathbf{0}$$

$$\Rightarrow -2v_1 + 2v_2 = 0 \Rightarrow v_1 = v_2$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Eigenvalues – example 1

- Consider now the other eigenvalue

$$(\mathbf{A} - 3\mathbf{I})\,\bar{\mathbf{v}} = \mathbf{0}$$

$$\Rightarrow \left(\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right) \bar{\mathbf{v}} = \mathbf{0}$$

$$\Rightarrow \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \bar{\mathbf{v}} = \mathbf{0}$$

Eigenvalues – example 1

- Which can be simplified as

$$\Rightarrow \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathbf{0}$$

$$\Rightarrow 2v_1 + 2v_2 = 0 \Rightarrow v_1 = -v_2$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = k \begin{bmatrix} 1 \\ -1 \end{bmatrix} \sim \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Eigenvalues – example 1

- Therefore, the **eigenvalues** and **eigenvectors** are

$$\lambda_1 = 7, \bar{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 3, \bar{\mathbf{v}}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Eigenvalues – example 2

- Consider the 3×3 matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 \\ -1 & 0 & -1 \\ 2 & 4 & 4 \end{bmatrix}$$

- Its characteristic polynomial is

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 3-\lambda & 2 & 1 \\ -1 & -\lambda & -1 \\ 2 & 4 & 4-\lambda \end{vmatrix}$$

Eigenvalues – example 2

- Expanding along the first row

$$
\begin{aligned}
|\mathbf{A}-\lambda\mathbf{I}| &= (3-\lambda)\left((-\lambda)(4-\lambda)+4\right)-2(-4+\lambda+2)+1(-4+2\lambda)\\
&= (3-\lambda)\left(\lambda^2-4\lambda+4\right)-2(\lambda-2)+2(\lambda-2)\\
&= -\lambda^3+7\lambda^2-16\lambda+12=0\\
&\Rightarrow -(\lambda-2)^2(\lambda-3)=0\\
&\Rightarrow \lambda=2,3
\end{aligned}
$$

Eigenvalues – example 2

- Thus there is a **double eigenvalue** $\lambda = 2$.
- This is termed the **algebraic multiplicity** of the eigenvalue (here 2)

Eigenvalues – example 2

- Consider now $\lambda = 2$. We determine the corresponding eigenvectors as

$$(\mathbf{A} - \lambda\mathbf{I})\,\bar{\mathbf{v}} = \mathbf{0}$$

$$\Rightarrow (\mathbf{A} - 2\mathbf{I})\,\bar{\mathbf{v}} = \mathbf{0}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$$

$$\Rightarrow x + 2y + z = 0$$

Eigenvalues – example 2

- Therefore, the **eigenvectors** are of the form

$$\begin{bmatrix} -2y - z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Eigenvalues – example 2

- Corresponding to the eigenvalue $\lambda = 3$, the eigenvector is found from

$$(\mathbf{A} - 3\mathbf{I})\,\bar{\mathbf{v}} = \mathbf{0}$$

$$\Rightarrow \begin{bmatrix} 0 & 2 & 1 \\ -1 & -3 & -1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$$

$$\Rightarrow 2y + z = 0, x + 3y + z = 0$$

Eigenvalues – example 2

- Hence $z = -2y$ and $x = -y$. The **eigenvector** is of the form

$$\begin{bmatrix} x \\ -x \\ 2x \end{bmatrix} = x \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Eigenvalues – example 2

- Therefore, the matrix has 3 **eigenvectors**, which span $\mathbb{R}^3$

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

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Complete eigenspace

Distinct eigenvalues give independent eigenvectors



Complete eigenspace

- The eigenspace of an $n \times n$ **real matrix** is **complete**
  - if its eigenvectors span $\mathbb{R}^n$

Distinct eigenvalues

- If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are **distinct eigenvalues** of a matrix $\mathbf{A}$,
  - the corresponding eigenvectors are **linearly independent**
- Consider this to be true for $k-1$ eigenvectors
  - i.e. let any set of $k-1$ eigenvectors be linearly independent

Distinct eigenvalues

- Let $\bar{\mathbf{v}}_{i_1}, \bar{\mathbf{v}}_{i_2}, \ldots, \bar{\mathbf{v}}_{i_k}$ be **eigenvectors** such that $c_{i_j} \neq 0$ and

$$c_{i_1}\bar{\mathbf{v}}_{i_1} + \ldots + c_{i_k}\bar{\mathbf{v}}_{i_k} = \mathbf{0}$$

Distinct eigenvalues

- Then

$$\mathbf{A}\left(c_{i_1}\bar{\mathbf{v}}_{i_1} + \ldots + c_{i_k}\bar{\mathbf{v}}_{i_k}\right) = \mathbf{0}$$
$$\Rightarrow c_{i_1}\lambda_{i_1}\bar{\mathbf{v}}_{i_1} + \ldots + c_{i_k}\lambda_{i_k}\bar{\mathbf{v}}_{i_k} = \mathbf{0}$$

- Further, multiplying original equation by λ_{i_k}

$$\Rightarrow c_{i_1}\lambda_{i_k}\bar{\mathbf{v}}_{i_1} + \ldots + c_{i_k}\lambda_{i_k}\bar{\mathbf{v}}_{i_k} = \mathbf{0}$$

Distinct eigenvalues

- Subtracting one from other we obtain

$$c_{i_1}\left(\lambda_{i_1}-\lambda_{i_k}\right)\bar{\mathbf{v}}_{i_1}+\ldots+c_{i_{k-1}}\left(\lambda_{i_{k-1}}-\lambda_{i_k}\right)\bar{\mathbf{v}}_{i_{k-1}}=\mathbf{0}$$

- This is a **contradiction** as we assumed that any set of $k-1$ **eigenvectors** is linearly independent

Eigenvalue decomposition

- Therefore, if the $n \times n$ real matrix $\mathbf{A}$ has n **distinct real eigenvalues** $\lambda_1, \lambda_2, \ldots, \lambda_n$,
  - then the corresponding real eigenvectors $\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \ldots, \bar{\mathbf{v}}_n$ form a basis of $\mathbb{R}^n$.
- Hence, it is **complete**

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Eigenvalue decomposition

Diagonalizing a matrix with its eigenvectors



Eigenvalue decomposition

- Consider an $n \times n$ real matrix $\mathbf{A}$ with eigenvectors $\bar{\mathbf{v}}_1$, $\bar{\mathbf{v}}_2$, …, $\bar{\mathbf{v}}_n$ that form a basis of $\mathbb{R}^n$

$$\begin{aligned}
\mathbf{A}\bar{\mathbf{v}}_1 &= \lambda_1 \bar{\mathbf{v}}_1 \\
\mathbf{A}\bar{\mathbf{v}}_2 &= \lambda_2 \bar{\mathbf{v}}_2 \\
&\vdots \\
\mathbf{A}\bar{\mathbf{v}}_n &= \lambda_n \bar{\mathbf{v}}_n
\end{aligned}$$

Eigenvalue decomposition

- Define the matrix of **eigenvectors** $\mathbf{V}$ as

$$\mathbf{V} = \begin{bmatrix} | & | & & | \\ \bar{\mathbf{v}}_1 & \bar{\mathbf{v}}_2 & \dots & \bar{\mathbf{v}}_n \\ | & | & & | \end{bmatrix}$$

Eigenvalue decomposition

- Define the **diagonal matrix** of eigenvalues $\mathbf{\Lambda}$ as

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

Eigenvalue decomposition

- Therefore, we have

$$\mathbf{A}\begin{bmatrix} | & | & & | \\ \bar{\mathbf{v}}_1 & \bar{\mathbf{v}}_2 & \dots & \bar{\mathbf{v}}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \bar{\mathbf{v}}_1 & \bar{\mathbf{v}}_2 & \dots & \bar{\mathbf{v}}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \dots & \\ & & & \lambda_n \end{bmatrix}$$

Eigenvalue decomposition

- This implies that

$$\mathbf{AV} = \mathbf{V}\boldsymbol{\Lambda}$$
$$\Rightarrow \mathbf{A} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}$$

- This is termed the **eigenvalue decomposition**.

Eigenvalue decomposition

- Furthermore, we also have

$$\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

- Such a matrix $\mathbf{A}$ is said to be diagonalizable.
  - Note that not all matrices are diagonalizable
  - We will explore this later

Eigenvalue decomposition – example

- Consider the 3×3 matrix $\mathbf{A}$ of example 2

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 \\ -1 & 0 & -1 \\ 2 & 4 & 4 \end{bmatrix}$$

Eigenvalue decomposition – example

- As seen earlier, the **eigenvectors** are

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Eigenvalue decomposition – example

- The **eigenvector matrix** and its **inverse** are

$$\mathbf{V} = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\mathbf{V}^{-1} = \begin{bmatrix} 1 & 3 & 1 \\ -2 & -4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

Eigenvalue decomposition – example

- The **eigenvalue decomposition** is

$$\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$$

$$= \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ -2 & -4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

Eigenvalue decomposition – example

- It can then be checked that

$$\mathbf{V}^{-1}\mathbf{A}\mathbf{V}$$

$$= \begin{bmatrix} 1 & 3 & 1 \\ -2 & -4 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ -1 & 0 & -1 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \mathbf{\Lambda}$$

Summary

- λ is an **eigenvalue** of $\mathbf{A}$ if $\mathbf{A}\bar{\mathbf{v}} = \lambda\bar{\mathbf{v}}$ for some $\bar{\mathbf{v}} \neq \mathbf{0}$, the **eigenvector**
- Eigenvalues are the roots of the characteristic equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$
- Eigenvectors for λ span the null space of $\mathbf{A} - \lambda\mathbf{I}$; a repeated root may give several
- Eigenvectors of distinct eigenvalues are linearly independent
- With n independent eigenvectors, $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$ and $\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \mathbf{\Lambda}$ is diagonal



Thank You!

Questions?