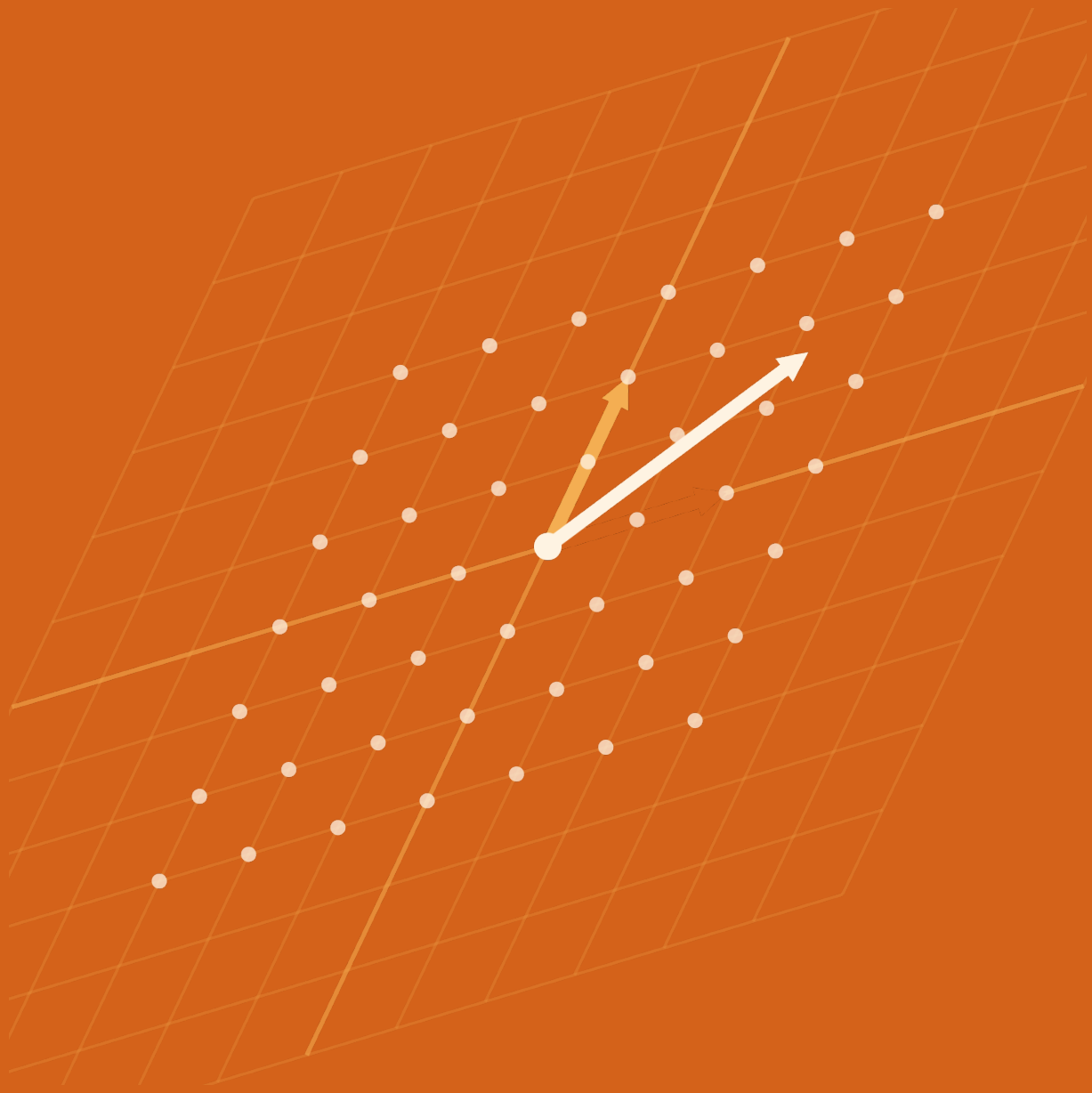
LINEAR ALGEBRA FOR AIML AND 6G

Inner Product Spaces: Orthogonal Sets and Gram–Schmidt

Lecture 14

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Outline

1. **Orthogonal complement**
   Vectors orthogonal to every vector of a set
2. **Orthogonal sets and bases**
   Orthogonal and orthonormal sets, coordinates in an orthogonal basis
3. **Projection**
   Component of a vector along another vector, and the remainder
4. **Gram-Schmidt orthogonalization**
   Building an orthogonal basis from any basis

01

Orthogonal complement

Vectors orthogonal to every vector of a set



Orthogonal complement

- Let $\mathcal{S}$ be a subset of an inner product space $\mathcal{V}$
- The **orthogonal complement** of $\mathcal{S}$, denoted by $\mathcal{S}^{\perp}$ consists of vectors $\bar{\mathbf{v}}$ that are **orthogonal** to every vector in $\mathcal{S}$

$$\mathcal{S}^{\perp} = \{\bar{\mathbf{v}} \in \mathcal{V} : \langle \bar{\mathbf{v}}, \bar{\mathbf{u}} \rangle = 0 \text{ for every } \bar{\mathbf{u}} \in \mathcal{S}\}$$

Orthogonal complement

- In particular for $\mathcal{S} = \{\bar{\mathbf{u}}\}$

$$\bar{\mathbf{u}}^{\perp} = \{\bar{\mathbf{v}} \in \mathcal{V} : \langle \bar{\mathbf{v}}, \bar{\mathbf{u}} \rangle = 0\}$$

Orthogonal complement

- $\mathcal{S}^{\perp}$ is a **vector space**
- $\mathbf{0} \in \mathcal{S}^{\perp}$ since

$$\langle \mathbf{0}, \bar{\mathbf{u}} \rangle = 0 \text{ for every } \bar{\mathbf{u}} \in \mathcal{S}$$

Orthogonal complement

- Further if $\bar{\mathbf{v}}, \bar{\mathbf{w}} \in \mathcal{S}^{\perp}$

$$\langle \bar{\mathbf{v}}, \bar{\mathbf{u}} \rangle = 0 \text{ for every } \bar{\mathbf{u}} \in \mathcal{S}$$
$$\langle \bar{\mathbf{w}}, \bar{\mathbf{u}} \rangle = 0 \text{ for every } \bar{\mathbf{u}} \in \mathcal{S}$$

- Then

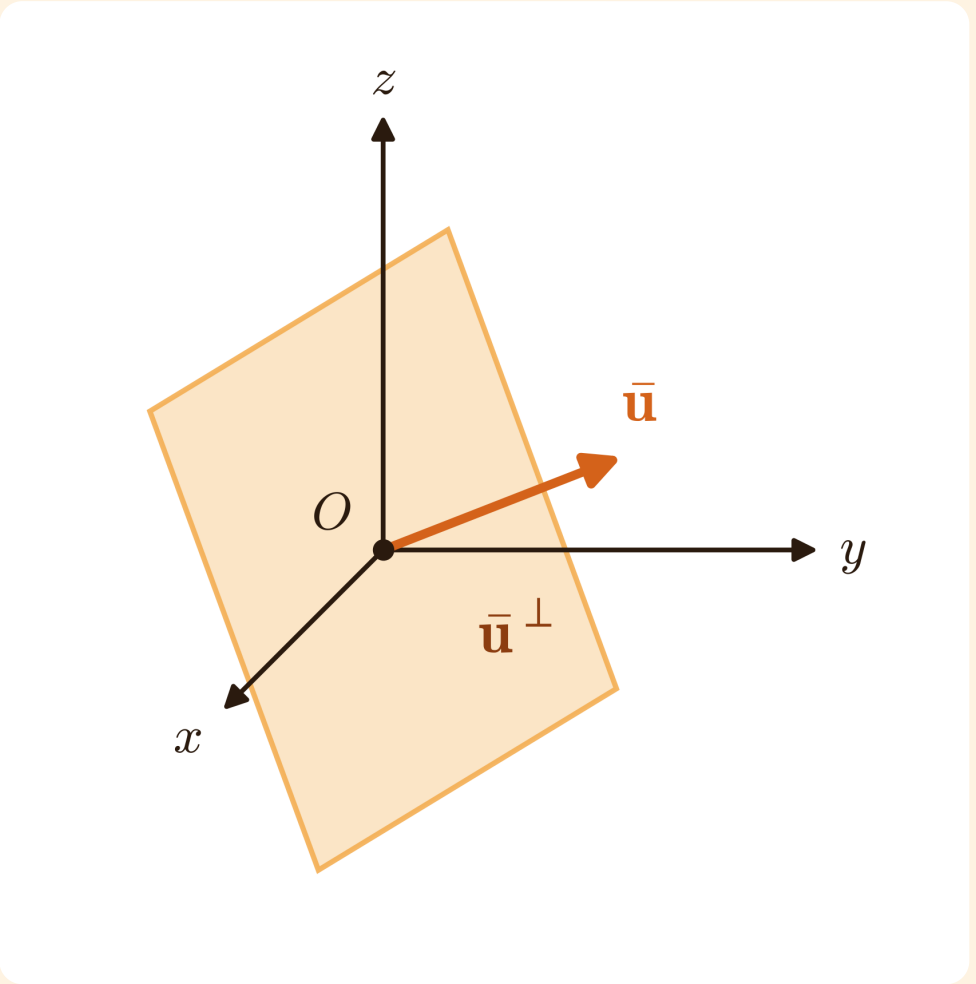
$$\langle \alpha\bar{\mathbf{v}} + \beta\bar{\mathbf{w}}, \bar{\mathbf{u}} \rangle = \alpha \langle \bar{\mathbf{v}}, \bar{\mathbf{u}} \rangle + \beta \langle \bar{\mathbf{w}}, \bar{\mathbf{u}} \rangle$$
$$= 0 \text{ for every } \bar{\mathbf{u}} \in \mathcal{S}$$

Orthogonal complement

- Therefore $\alpha\bar{\mathbf{v}} + \beta\bar{\mathbf{w}} \in \mathcal{S}^{\perp}$
- It follows that $\mathcal{S}^{\perp}$ is a **vector space**

Geometry

- Suppose $\bar{\mathbf{u}}$ is a nonzero vector in $\mathbb{R}^3$. Then there is a **geometrical description** of $\bar{\mathbf{u}}^{\perp}$
- $\bar{\mathbf{u}}^{\perp}$ is the plane in $\mathbb{R}^3$ through the origin O and **perpendicular** to the vector $\bar{\mathbf{u}}$



Homogeneous system

- Let $\mathcal{W}$ be the solution space of an $m \times n$ **homogeneous system** $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$
- Specifically, each solution vector $\bar{\mathbf{x}} = [x_1, x_2, \ldots, x_n]^T$ is **orthogonal** to each row of $\mathbf{A}$

Homogeneous system

- Hence, $\mathcal{W}$ is the **orthogonal complement** of the **row space** of $\mathbf{A}$.
- This is the **null space** of $\mathbf{A}$

Homogeneous system

- Find a basis for the subspace $\bar{u}^{\perp}$ of $\mathbb{R}^3$, where $\bar{u} = [1\ -2\ 5]^T$
- $\bar{u}^{\perp}$ consists of vectors $\bar{x} = [x\ y\ z]^T$ with $\langle \bar{x}, \bar{u} \rangle = 0$

$$\begin{bmatrix} 1 & -2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x - 2y + 5z = 0$$

Homogeneous system

- This can be simplified as below

$$x - 2y + 5z = 0$$

$$\Rightarrow x = 2y - 5z$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2y - 5z \\ y \\ z \end{bmatrix}$$

$$= y \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$$

Homogeneous system

- Therefore, basis for the **subspace** $\bar{\mathbf{u}}^{\perp}$ of $\mathbb{R}^3$ is

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$$

02

Orthogonal sets and bases

Orthogonal and orthonormal sets, coordinates in an orthogonal basis



Orthogonal set

- Consider a set $\mathcal{S} = \{\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \ldots, \bar{\mathbf{u}}_r\}$.
- $\mathcal{S}$ is termed **orthogonal** if each pair of distinct vectors in $\mathcal{S}$ is orthogonal

$$\langle \bar{\mathbf{u}}_i, \bar{\mathbf{u}}_j \rangle = 0, i \neq j$$

Orthonormal set

- $\mathcal{S}$ is termed **orthonormal** if it is **orthogonal** and each vector has **unit length**

$$\langle \bar{\mathbf{u}}_i, \bar{\mathbf{u}}_j \rangle = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Orthogonal set

- Suppose $\mathcal{S}$ is an **orthogonal set** of nonzero vectors.
- Then $\mathcal{S}$ is **linearly independent**.

Orthogonal set

- Suppose $\{\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \ldots, \bar{\mathbf{u}}_r\}$ is an **orthogonal set** of vectors.
- Then

$$\left\|\bar{\mathbf{u}}_1 + \bar{\mathbf{u}}_2 + \ldots + \bar{\mathbf{u}}_r\right\|^2$$
$$= \left\|\bar{\mathbf{u}}_1\right\|^2 + \left\|\bar{\mathbf{u}}_2\right\|^2 + \ldots + \left\|\bar{\mathbf{u}}_r\right\|^2$$

Orthogonal set

- This follows since $\langle \bar{\mathbf{u}}_i, \bar{\mathbf{u}}_j \rangle = 0$ for $i \neq j$

$$\left\| \bar{\mathbf{u}}_1 + \bar{\mathbf{u}}_2 + \ldots + \bar{\mathbf{u}}_r \right\|^2$$
$$= \left\langle \sum_{i=1}^{r} \bar{\mathbf{u}}_i, \sum_{j=1}^{r} \bar{\mathbf{u}}_j \right\rangle$$

Orthogonal set

- Separating the terms with $i = j$ and $i \neq j$

$$= \sum_{i=1}^{r} \langle \bar{\mathbf{u}}_i, \bar{\mathbf{u}}_i \rangle + \underbrace{\sum_{i=1}^{r} \sum_{\substack{j=1 \\ j \neq i}}^{r} \langle \bar{\mathbf{u}}_i, \bar{\mathbf{u}}_j \rangle}_{0}$$

$$= \sum_{i=1}^{r} \|\bar{\mathbf{u}}_i\|^2$$

Orthogonal set – example

- Consider the usual basis in $\mathbb{R}^3$

$$\bar{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \bar{\mathbf{e}}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \bar{\mathbf{e}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Orthogonal set – example

- It can be seen that

$$\langle \bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2 \rangle , \langle \bar{\mathbf{e}}_1, \bar{\mathbf{e}}_3 \rangle , \langle \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3 \rangle = 0$$

$$\langle \bar{\mathbf{e}}_1, \bar{\mathbf{e}}_1 \rangle , \langle \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_2 \rangle , \langle \bar{\mathbf{e}}_3, \bar{\mathbf{e}}_3 \rangle = 1$$

- Thus $\bar{\mathbf{e}}_1$, $\bar{\mathbf{e}}_2$, $\bar{\mathbf{e}}_3$ form an **orthonormal basis** of $\mathbb{R}^3$

Orthogonal set of functions

- Let $\mathcal{V} = C[-\pi,\pi]$ be the **vector space** of continuous functions on the interval $-\pi \leq t \leq \pi$
- The inner product is defined as

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t)\, g(t)\, dt$$

Orthogonal set of functions

- A classical example of an **orthogonal set** in $\mathcal{V}$ is $\{1,\cos\, t,\cos\, 2t,\cos\, 3t,\ldots,\sin\, t,\sin\, 2t,\sin\, 3t,\ldots\}$

$$\langle \sin kt, \cos lt \rangle = \int_{-\pi}^{\pi} \sin kt \cos lt dt$$

Orthogonal set of functions

- For $k \neq l$, using $2\sin kt\cos lt = \sin(k+l)t + \sin(k-l)t$

$$\int_{-\pi}^{\pi} \sin kt \cos ltdt$$

$$= -\frac{1}{2}\left[\frac{\cos(k+l)t}{k+l} + \frac{\cos(k-l)t}{k-l}\right]_{-\pi}^{\pi} = 0$$

Linear combination

- Let $\{\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \ldots, \bar{\mathbf{u}}_n\}$ be an **orthogonal basis** of $\mathcal{V}$
- Then, for any $\bar{\mathbf{v}} \in \mathcal{V}$

$$\bar{\mathbf{v}} = a_1\bar{\mathbf{u}}_1 + a_2\bar{\mathbf{u}}_2 + \ldots + a_n\bar{\mathbf{u}}_n$$

- where

$$a_i = \frac{\langle \bar{\mathbf{v}}, \bar{\mathbf{u}}_i \rangle}{\langle \bar{\mathbf{u}}_i, \bar{\mathbf{u}}_i \rangle}$$

Linear combination

- This can be seen as follows

$$\begin{aligned}
&\langle \bar{\mathbf{v}}, \bar{\mathbf{u}}_i \rangle \\
&= a_1 \langle \bar{\mathbf{u}}_1, \bar{\mathbf{u}}_i \rangle + a_2 \langle \bar{\mathbf{u}}_2, \bar{\mathbf{u}}_i \rangle + \ldots + a_n \langle \bar{\mathbf{u}}_n, \bar{\mathbf{u}}_i \rangle \\
&= a_i \langle \bar{\mathbf{u}}_i, \bar{\mathbf{u}}_i \rangle \\
&\Rightarrow a_i = \frac{\langle \bar{\mathbf{v}}, \bar{\mathbf{u}}_i \rangle}{\langle \bar{\mathbf{u}}_i, \bar{\mathbf{u}}_i \rangle}
\end{aligned}$$

03

Projection

Component of a vector along another vector, and the remainder



Projection

- Let $\mathcal{V}$ be an **inner product space**
- The **projection** of $\bar{\mathbf{v}}$ along $\bar{\mathbf{w}}$ is given as

$$\frac{\langle \bar{\mathbf{v}}, \bar{\mathbf{w}} \rangle}{\langle \bar{\mathbf{w}}, \bar{\mathbf{w}} \rangle} \bar{\mathbf{w}}$$

Projection

- The component of $\bar{\mathbf{v}}$ orthogonal to $\bar{\mathbf{w}}$ is given as

$$\bar{\mathbf{v}} - \frac{\langle \bar{\mathbf{v}}, \bar{\mathbf{w}} \rangle}{\langle \bar{\mathbf{w}}, \bar{\mathbf{w}} \rangle} \bar{\mathbf{w}}$$

Projection

- Note that the remainder is orthogonal to $\bar{\mathbf{w}}$

$$\left\langle \bar{\mathbf{v}} - \frac{\langle \bar{\mathbf{v}}, \bar{\mathbf{w}} \rangle}{\langle \bar{\mathbf{w}}, \bar{\mathbf{w}} \rangle} \bar{\mathbf{w}}, \bar{\mathbf{w}} \right\rangle = \langle \bar{\mathbf{v}}, \bar{\mathbf{w}} \rangle - \frac{\langle \bar{\mathbf{v}}, \bar{\mathbf{w}} \rangle}{\langle \bar{\mathbf{w}}, \bar{\mathbf{w}} \rangle} \langle \bar{\mathbf{w}}, \bar{\mathbf{w}} \rangle$$

$$= \langle \bar{\mathbf{v}}, \bar{\mathbf{w}} \rangle - \langle \bar{\mathbf{v}}, \bar{\mathbf{w}} \rangle = 0$$

Projection on a subspace

- Suppose $\mathcal{W} = \{\bar{\mathbf{w}}_1, \bar{\mathbf{w}}_2, \ldots, \bar{\mathbf{w}}_r\}$ form an **orthogonal set**
- The **projection** of $\bar{\mathbf{v}}$ on the space spanned by $\mathcal{W}$, and the component orthogonal to it, are

$$\frac{\langle \bar{\mathbf{v}}, \bar{\mathbf{w}}_1 \rangle}{\langle \bar{\mathbf{w}}_1, \bar{\mathbf{w}}_1 \rangle} \bar{\mathbf{w}}_1 + \ldots + \frac{\langle \bar{\mathbf{v}}, \bar{\mathbf{w}}_r \rangle}{\langle \bar{\mathbf{w}}_r, \bar{\mathbf{w}}_r \rangle} \bar{\mathbf{w}}_r$$

$$\bar{\mathbf{v}} - \frac{\langle \bar{\mathbf{v}}, \bar{\mathbf{w}}_1 \rangle}{\langle \bar{\mathbf{w}}_1, \bar{\mathbf{w}}_1 \rangle} \bar{\mathbf{w}}_1 - \ldots - \frac{\langle \bar{\mathbf{v}}, \bar{\mathbf{w}}_r \rangle}{\langle \bar{\mathbf{w}}_r, \bar{\mathbf{w}}_r \rangle} \bar{\mathbf{w}}_r$$

04

Gram-Schmidt orthogonalization

Building an orthogonal basis from any basis



Gram-Schmidt

- Suppose $\{\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \ldots, \bar{\mathbf{v}}_n\}$ form a basis for an inner product space $\mathcal{V}$

Gram-Schmidt

- One can use this to construct an **orthogonal basis** $\{\bar{\mathbf{w}}_1, \bar{\mathbf{w}}_2, \ldots, \bar{\mathbf{w}}_n\}$ as follows

$$\bar{\mathbf{w}}_1 = \bar{\mathbf{v}}_1$$

$$\bar{\mathbf{w}}_2 = \bar{\mathbf{v}}_2 - \frac{\langle \bar{\mathbf{v}}_2, \bar{\mathbf{w}}_1 \rangle}{\langle \bar{\mathbf{w}}_1, \bar{\mathbf{w}}_1 \rangle} \bar{\mathbf{w}}_1$$

$$\bar{\mathbf{w}}_3 = \bar{\mathbf{v}}_3 - \frac{\langle \bar{\mathbf{v}}_3, \bar{\mathbf{w}}_1 \rangle}{\langle \bar{\mathbf{w}}_1, \bar{\mathbf{w}}_1 \rangle} \bar{\mathbf{w}}_1 - \frac{\langle \bar{\mathbf{v}}_3, \bar{\mathbf{w}}_2 \rangle}{\langle \bar{\mathbf{w}}_2, \bar{\mathbf{w}}_2 \rangle} \bar{\mathbf{w}}_2$$

Gram-Schmidt

- The expression for $\bar{\mathbf{w}}_n$ is given as

$$\bar{\mathbf{w}}_n = \bar{\mathbf{v}}_n - \frac{\langle \bar{\mathbf{v}}_n, \bar{\mathbf{w}}_1 \rangle}{\langle \bar{\mathbf{w}}_1, \bar{\mathbf{w}}_1 \rangle} \bar{\mathbf{w}}_1 - \frac{\langle \bar{\mathbf{v}}_n, \bar{\mathbf{w}}_2 \rangle}{\langle \bar{\mathbf{w}}_2, \bar{\mathbf{w}}_2 \rangle} \bar{\mathbf{w}}_2 - \dots - \frac{\langle \bar{\mathbf{v}}_n, \bar{\mathbf{w}}_{n-1} \rangle}{\langle \bar{\mathbf{w}}_{n-1}, \bar{\mathbf{w}}_{n-1} \rangle} \bar{\mathbf{w}}_{n-1}$$

- This is termed **Gram–Schmidt orthogonalization** process

Gram-Schmidt

- Note that the component of $\bar{\mathbf{v}}_k$ along $\bar{\mathbf{w}}_i$ is

$$\frac{\langle \bar{\mathbf{v}}_k, \bar{\mathbf{w}}_i \rangle}{\langle \bar{\mathbf{w}}_i, \bar{\mathbf{w}}_i \rangle} \bar{\mathbf{w}}_i$$

Gram-Schmidt

- Removing all components along $\bar{\mathbf{w}}_1, \bar{\mathbf{w}}_2, \ldots, \bar{\mathbf{w}}_{k-1}$ we obtain the remainder $\bar{\mathbf{w}}_k$
- This is therefore orthogonal to $\bar{\mathbf{w}}_1, \bar{\mathbf{w}}_2, \ldots, \bar{\mathbf{w}}_{k-1}$

Gram-Schmidt – example

- Find an orthogonal basis and then an **orthonormal basis** for the subspace $\mathcal{U}$ of $\mathbb{R}^4$ spanned by

$$\bar{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \bar{\mathbf{v}}_2 = \begin{bmatrix} 1 \\ 4 \\ 2 \\ 5 \end{bmatrix}, \bar{\mathbf{v}}_3 = \begin{bmatrix} 1 \\ -3 \\ 0 \\ -2 \end{bmatrix}$$

Gram-Schmidt – example

- Set

$$\bar{\mathbf{w}}_1 = \bar{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Gram-Schmidt – example

- Next $\bar{\mathbf{w}}_2$ is calculated as

$$\bar{\mathbf{w}}_2 = \bar{\mathbf{v}}_2 - \frac{\langle \bar{\mathbf{v}}_2, \bar{\mathbf{w}}_1 \rangle}{\langle \bar{\mathbf{w}}_1, \bar{\mathbf{w}}_1 \rangle} \bar{\mathbf{w}}_1$$

$$= \begin{bmatrix} 1 \\ 4 \\ 2 \\ 5 \end{bmatrix} - \frac{12}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 2 \end{bmatrix}$$

Gram-Schmidt – example

- Now $\bar{\mathbf{w}}_3$ is computed as

$$\bar{\mathbf{w}}_3 = \bar{\mathbf{v}}_3 - \frac{\langle \bar{\mathbf{v}}_3, \bar{\mathbf{w}}_1 \rangle}{\langle \bar{\mathbf{w}}_1, \bar{\mathbf{w}}_1 \rangle} \bar{\mathbf{w}}_1 - \frac{\langle \bar{\mathbf{v}}_3, \bar{\mathbf{w}}_2 \rangle}{\langle \bar{\mathbf{w}}_2, \bar{\mathbf{w}}_2 \rangle} \bar{\mathbf{w}}_2$$

$$= \begin{bmatrix} 1 \\ -3 \\ 0 \\ -2 \end{bmatrix} - \frac{-4}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{-9}{10} \begin{bmatrix} -2 \\ 1 \\ -1 \\ 2 \end{bmatrix}$$

Gram-Schmidt – example

- This is further simplified as

$$\bar{\mathbf{w}}_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \\ -1 \end{bmatrix} + \frac{9}{10} \begin{bmatrix} -2 \\ 1 \\ -1 \\ 2 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 2 \\ -11 \\ 1 \\ 8 \end{bmatrix}$$

Gram-Schmidt – example

- One can normalize $\bar{\mathbf{w}}_1$, $\bar{\mathbf{w}}_2$ and $\bar{\mathbf{w}}_3$ to find an **orthonormal basis**

$$\tilde{\mathbf{w}}_1 = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \tilde{\mathbf{w}}_2 = \frac{1}{\sqrt{10}}\begin{bmatrix}-2\\1\\-1\\2\end{bmatrix}, \tilde{\mathbf{w}}_3 = \frac{1}{\sqrt{190}}\begin{bmatrix}2\\-11\\1\\8\end{bmatrix}$$

Gram-Schmidt – polynomials

- Let $\mathcal{V}$ be the vector space of polynomials with **inner product**

$$\langle f, g \rangle = \int_{-1}^{1} f(t)\, g(t)\, dt$$

Gram-Schmidt – polynomials

- Apply the **Gram-Schmidt orthogonalization** process to $\{1, t, t^2, t^3\}$ to find an **orthogonal basis** $\{f_0, f_1, f_2, f_3\}$

Gram-Schmidt – polynomials

- Set $f_0 = 1$
- f_1 can be evaluated as

$$f_1 = t - \frac{\langle t, 1 \rangle}{\langle 1, 1 \rangle} 1$$

$$= t - \frac{1}{2} \int_{-1}^{1} t dt = t - 0 = t$$

Gram-Schmidt – polynomials

- f_2 can be evaluated as

$$f_2 = t^2 - \frac{\langle t^2, 1\rangle}{\langle 1, 1\rangle}1 - \frac{\langle t^2, t\rangle}{\langle t, t\rangle}t$$

$$= t^2 - \frac{1}{2}\int_{-1}^{1} t^2 dt - \frac{t}{2/3}\int_{-1}^{1} t^3 dt$$

$$= t^2 - \frac{\frac{2}{3}}{2} = t^2 - \frac{1}{3} \sim 3t^2 - 1$$

Gram-Schmidt – polynomials

- f_3 can be evaluated as

$$f_3 = t^3 - \frac{\langle t^3, 1\rangle}{\langle 1, 1\rangle}1 - \frac{\langle t^3, t\rangle}{\langle t, t\rangle}t - \frac{\langle t^3, 3t^2-1\rangle}{\langle 3t^2-1, 3t^2-1\rangle}\left(3t^2-1\right)$$

$$= t^3 - 0 - \frac{t}{2/3}\int_{-1}^{1} t^4 dt - 0$$

$$= t^3 - \frac{\frac{2}{5}}{\frac{2}{3}}t = t^3 - \frac{3}{5}t \sim 5t^3 - 3t$$

Gram-Schmidt – polynomials

- Renormalizing polynomials such that $p(1) = 1$ yields

$$1, t, \frac{1}{2}\left(3t^2 - 1\right), \frac{1}{2}\left(5t^3 - 3t\right)$$

Summary

- $\mathcal{S}^{\perp}$ contains every vector orthogonal to all of $\mathcal{S}$; it is a vector space
- The solutions of $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$ form the orthogonal complement of the row space of $\mathbf{A}$
- Nonzero orthogonal vectors are linearly independent, and $\|\sum_i \bar{\mathbf{u}}_i\|^2 = \sum_i \|\bar{\mathbf{u}}_i\|^2$
- In an orthogonal basis, $\bar{\mathbf{v}} = \Sigma a_i \bar{\mathbf{u}}_i$ with $a_i = \langle \bar{\mathbf{v}}, \bar{\mathbf{u}}_i \rangle / \langle \bar{\mathbf{u}}_i, \bar{\mathbf{u}}_i \rangle$
- Projection of $\bar{\mathbf{v}}$ along $\bar{\mathbf{w}}$ is $(\langle \bar{\mathbf{v}}, \bar{\mathbf{w}} \rangle / \langle \bar{\mathbf{w}}, \bar{\mathbf{w}} \rangle)\bar{\mathbf{w}}$; the remainder is orthogonal to $\bar{\mathbf{w}}$
- **Gram–Schmidt** subtracts these projections to turn any basis into an orthogonal basis



Thank You!

Questions?