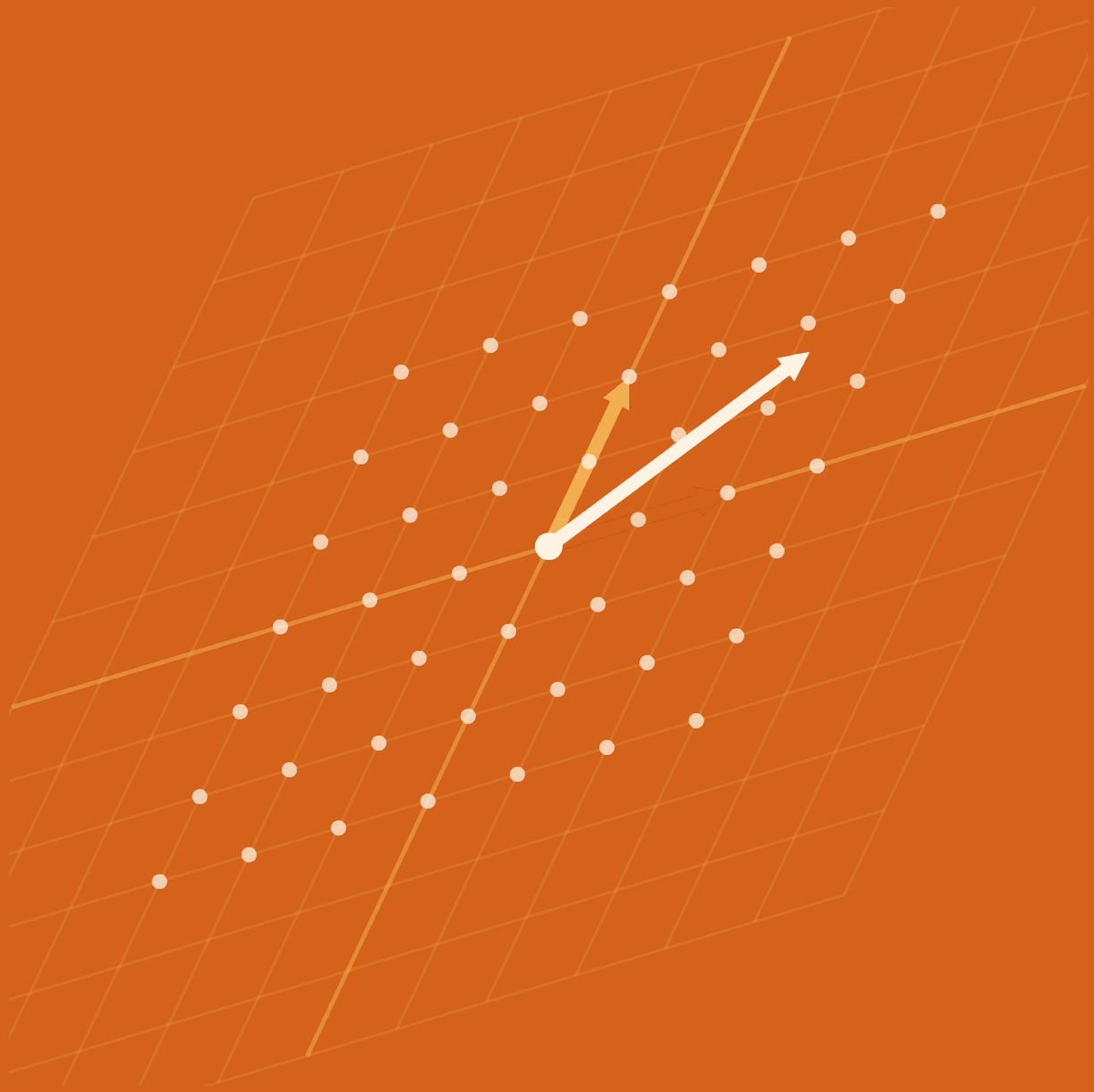
LINEAR ALGEBRA FOR AIML AND 6G

Inner product spaces: Definition and Norm



Lecture 13

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Outline

1. **Inner product spaces**
   Axioms, norm, and examples on R^n, functions, matrices and sequences

2. **Cauchy–Schwarz inequality**
   An upper bound on the inner product

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4. **Angle between vectors**
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01

Inner product spaces

Axioms, norm, and examples on R^n, functions, matrices and sequences



Inner product space

- Let $\mathcal{V}$ be a real vector space.
- Suppose to each pair of vectors $\bar{u}, \bar{v} \in \mathcal{V}$ there is assigned a **real number**, denoted by $\langle \bar{u}, \bar{v} \rangle$

Inner product space

- This function is called a (real) **inner product** on $\mathcal{V}$ if it satisfies the following axioms:

- **Axiom 1: Linear property**:

$$\langle a\bar{\mathbf{u}}_1 + b\bar{\mathbf{u}}_2, \bar{\mathbf{v}} \rangle = a \langle \bar{\mathbf{u}}_1, \bar{\mathbf{v}} \rangle + b \langle \bar{\mathbf{u}}_2, \bar{\mathbf{v}} \rangle$$

- **Axiom 2: Symmetric property**

$$\langle \bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle = \langle \bar{\mathbf{v}}, \bar{\mathbf{u}} \rangle$$

Inner product space

- **Axiom 3: Positive definite property**

$$\langle \bar{\mathbf{u}}, \bar{\mathbf{u}} \rangle \geq 0$$

- $\langle \bar{\mathbf{u}}, \bar{\mathbf{u}} \rangle = 0$ if and only if $\bar{\mathbf{u}} = \mathbf{0}$

Inner product space

- Using **linearity** and **symmetry** property it follows that

$$\begin{aligned}\langle \bar{\mathbf{u}}, c\bar{\mathbf{v}}_1 + d\bar{\mathbf{v}}_2 \rangle &= \langle c\bar{\mathbf{v}}_1 + d\bar{\mathbf{v}}_2, \bar{\mathbf{u}} \rangle \\ &= c\langle \bar{\mathbf{v}}_1, \bar{\mathbf{u}} \rangle + d\langle \bar{\mathbf{v}}_2, \bar{\mathbf{u}} \rangle \\ &= c\langle \bar{\mathbf{u}}, \bar{\mathbf{v}}_1 \rangle + d\langle \bar{\mathbf{u}}, \bar{\mathbf{v}}_2 \rangle\end{aligned}$$

Inner product space

- Since $\langle \bar{\mathbf{u}}, \bar{\mathbf{u}} \rangle \geq 0$, one can define the norm as

$$\|\bar{\mathbf{u}}\| = \sqrt{\langle \bar{\mathbf{u}}, \bar{\mathbf{u}} \rangle}$$

$$\|\bar{\mathbf{u}}\|^2 = \langle \bar{\mathbf{u}}, \bar{\mathbf{u}} \rangle$$

- The norm is also called as the length of $\bar{\mathbf{u}}$

Inner product space-Ex.

- **Euclidean** n-space $\mathbb{R}^n$
- Consider the vectors

$$\bar{\mathbf{u}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \bar{\mathbf{v}} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Inner product space-Ex.

- The inner product $\langle \bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle$ is

$$\langle \bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle = \bar{\mathbf{u}} \cdot \bar{\mathbf{v}} = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$$

$$\langle \bar{\mathbf{u}}, \bar{\mathbf{u}} \rangle = a_1^2 + a_2^2 + \ldots + a_n^2$$

$$\|\bar{\mathbf{u}}\| = \sqrt{\langle \bar{\mathbf{u}}, \bar{\mathbf{u}} \rangle}$$

$$= \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}$$

Function space

- Denote by $\mathcal{C}[a,b]$ the set of all **continuous functions** on the interval $[a,b]$

- The **inner product** is defined as

$$\langle f, g \rangle = \int_a^b f(t)\, g(t)\, dt$$

Function space-Example

- Consider $f(t) = 2t + 1$ and $g(t) = t^3$

- The inner product can be evaluated as

$$\langle f, g \rangle = \int_0^1 (2t+1)t^3 \, dt$$

$$= \left[\frac{2}{5}t^5 + \frac{1}{4}t^4 \right]_0^1 = \frac{2}{5} + \frac{1}{4} = \frac{13}{20}$$

Function space-Example

- $\|f\|$ can be found as

Function space-Example

$$\|f\|^2 = \int_0^1 (2t+1)^2\, dt$$

$$= \int_0^1 (4t^2 + 4t + 1)\, dt$$

$$= \left[\frac{4}{3}t^3 + 2t^2 + t\right]_0^1 = \frac{13}{3}$$

$$\|f\| = \sqrt{\frac{13}{3}}$$

Function space-Example

- $\|g\|$ can be found as

$$\|g\|^2 = \int_0^1 t^6\, dt = \left[\frac{t^7}{7}\right]_0^1 = \frac{1}{7}$$

$$\|g\| = \sqrt{\frac{1}{7}}$$

Matrix space

- Let $\mathbb{R}^{m\times n}$ denote the vector space of all real $m \times n$ matrices
- An **inner product** is defined on $\mathbb{R}^{m\times n}$ as

$$\langle \mathbf{A}, \mathbf{B} \rangle = \operatorname{tr}\left(\mathbf{B}^T \mathbf{A}\right)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ij}$$

Matrix space

- The **norm** $\|\mathbf{A}\|$ is defined as

$$\langle \mathbf{A}, \mathbf{A} \rangle = \|\mathbf{A}\|^2$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2$$

- In particular $\langle \mathbf{A}, \mathbf{A} \rangle = \|\mathbf{A}\|^2$ is the **sum of the squares** of the entries of $\mathbf{A}$

Hilbert space of sequences

- Let $\mathcal{H}$ be the **set** of all infinite sequences

$$a_1, a_2, a_3, \ldots$$

- Satisfying

$$\sum_{i=1}^{\infty} |a_i|^2 = a_1^2 + a_2^2 + a_3^2 + \ldots < \infty$$

Hilbert space of sequences

- $\mathcal{H}$ is termed the Hilbert space of sequences.
- This is a vector space since if

$$\mathbf{a} = a_1, a_2, a_3, \ldots \in \mathcal{H}$$
$$\mathbf{b} = b_1, b_2, b_3, \ldots \in \mathcal{H}$$

- Then $\alpha\mathbf{a} + \beta\mathbf{b} \in \mathcal{H}$

Hilbert space of sequences

- This can be seen as follows

$$\alpha\mathbf{a} + \beta\mathbf{b} = \alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \ \ldots$$

Hilbert space of sequences

- Then

$$\sum_{i=1}^{\infty} |\alpha a_i + \beta b_i|^2$$

$$= \sum_{i=1}^{\infty} \alpha^2 a_i^2 + \beta^2 b_i^2 + 2\alpha\beta a_i b_i$$

$$\leq 2 \sum_{i=1}^{\infty} \alpha^2 a_i^2 + \beta^2 b_i^2 < \infty$$

$$\Rightarrow \alpha\mathbf{a} + \beta\mathbf{b} \in \mathcal{H}$$

02

Cauchy–Schwarz inequality

An upper bound on the inner product



Cauchy–Schwarz

- **Cauchy–Schwarz (CS)** or Schwarz inequality is

$$\langle \bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle^2 \leq \langle \bar{\mathbf{u}}, \bar{\mathbf{u}} \rangle \langle \bar{\mathbf{v}}, \bar{\mathbf{v}} \rangle$$

$$|\langle \bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle| \leq \|\bar{\mathbf{u}}\| \, \|\bar{\mathbf{v}}\|$$

CS inequality

- Consider the vectors

$$\bar{\mathbf{u}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \bar{\mathbf{v}} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

- From Cauchy-Schwarz inequality

$$\begin{aligned} &(a_1 b_1 + \ldots + a_n b_n)^2 \\ &\leq \left(a_1^2 + \ldots + a_n^2\right)\left(b_1^2 + \ldots + b_n^2\right) \end{aligned}$$

CS inequality

- Let f and g be continuous functions on the interval $[0,1]$

$$\left| \int_0^1 f(t)g(t)\,dt \right|^2 \leq \int_0^1 f^2(t)\,dt \int_0^1 g^2(t)\,dt$$

$$\langle f, g \rangle^2 \leq \|f\|^2 \|g\|^2$$

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Norm properties

Non-negativity, homogeneity and the triangle inequality



Norm properties

- Let $\mathcal{V}$ be an **inner product space**
- Then the norm in $\mathcal{V}$ satisfies the following properties

$$\|\bar{\mathbf{v}}\| \geq 0$$

- And $\|\bar{\mathbf{v}}\| = 0 \Leftrightarrow \bar{\mathbf{v}} = \mathbf{0}$

Norm properties

- Let $\mathcal{V}$ be an inner product space
- Then the norm in $\mathcal{V}$ satisfies the following properties

$$\|k\bar{\mathbf{v}}\| = |k| \, \|\bar{\mathbf{v}}\|$$

$$\|\bar{\mathbf{u}} + \bar{\mathbf{v}}\| \leq \|\bar{\mathbf{u}}\| + \|\bar{\mathbf{v}}\|$$

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Angle between vectors

Angle and orthogonality in an inner product space



Angle between vectors

- By **Cauchy–Schwarz inequality**

$$-\|\bar{\mathbf{u}}\|\,\|\bar{\mathbf{v}}\| \leq \langle \bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle \leq \|\bar{\mathbf{u}}\|\,\|\bar{\mathbf{v}}\|$$

$$\Rightarrow -1 \leq \frac{\langle \bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle}{\|\bar{\mathbf{u}}\|\,\|\bar{\mathbf{v}}\|} \leq 1$$

Angle between vectors

- Therefore, the angle θ between vectors $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ in an **inner product space** is defined as

$$\cos\,\theta = \frac{\langle \bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle}{\|\bar{\mathbf{u}}\|\,\|\bar{\mathbf{v}}\|}$$

Angle – Example

- Consider vectors $[3\ -1\ 2]^T$ and $[2\ 4\ 1]^T$

$$\cos\theta = \frac{6-4+2}{\sqrt{14}\sqrt{21}} = \frac{4}{\sqrt{14}\sqrt{21}}$$

Angle – Example

- Consider the functions $f(t) = 2t + 1$ and $g(t) = t^3$

- The angle θ between f and g is

$$\cos\theta = \frac{\langle f, g\rangle}{\|f\|\|g\|} = \frac{13/20}{\sqrt{13/3}\sqrt{1/7}} = \frac{\sqrt{273}}{20}$$

Orthogonality

- Let $\mathcal{V}$ be an inner product space
- The vectors $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ are **orthogonal** if

$$\langle \bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle = 0$$

$$\cos\,\theta = 0$$

Orthogonality – Example

- Consider $\bar{u} = [2\ 1\ -1]^T$, $\bar{v} = [1\ -3\ -1]^T$
- The dot product of $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ is

$$\langle \bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle = 2 - 3 + 1 = 0$$

- Therefore they are **orthogonal**

Orthogonality functions

- Consider $\sin t$ and $\cos t$ over the interval $[-\pi,\pi]$
- Their **inner product** is

$$\int_{-\pi}^{\pi} \sin t \cos t dt = \frac{1}{2}\sin^2 t\Big|_{-\pi}^{\pi} = 0$$

Summary

- Inner product: linear, symmetric and positive definite
- Norm: $\|\bar{\mathbf{u}}\| = \sqrt{\langle \bar{\mathbf{u}}, \bar{\mathbf{u}} \rangle}$
- Examples: dot product on $\mathbb{R}^n$, $\int fg\, dt$, $\mathrm{tr}(\mathbf{B}^T\mathbf{A})$, square-summable sequences
- Cauchy–Schwarz: $|\langle \bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle| \leq \|\bar{\mathbf{u}}\| \|\bar{\mathbf{v}}\|$
- Angle: $\cos\theta = \langle \bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle / \|\bar{\mathbf{u}}\| \|\bar{\mathbf{v}}\|$; orthogonal when $\langle \bar{\mathbf{u}}, \bar{\mathbf{v}} \rangle = 0$



Thank You!

Questions?