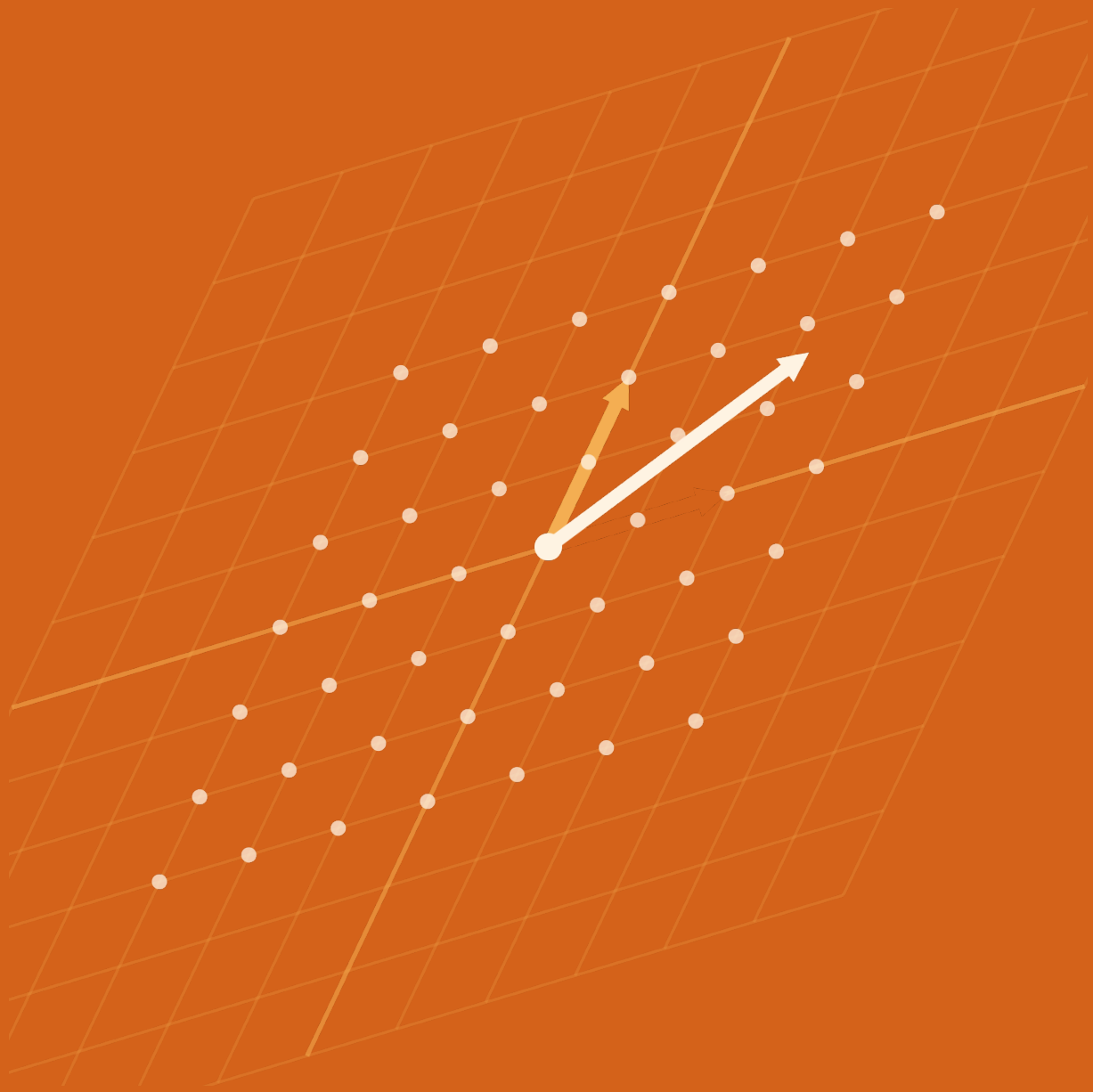
LINEAR ALGEBRA FOR AIML AND 6G

Vector spaces: Linear Independence, Bases



Lecture 12

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Outline

1. **Linear dependence and independence**
   When a combination of vectors can vanish

2. **Basis and dimension**
   Bases, dimension and their properties

3. **Rank of matrix**
   Row space, column space and the rank of a matrix

Outline

4 **Null space**

Solutions of Ax = 0 and the rank–nullity result

5 **Four fundamental subspaces**

Row, column and null spaces of A and its transpose

01

Linear dependence and independence

When a combination of vectors can vanish



Linear dependence (LD)

- Let $\mathcal{V}$ be a vector space over a field $\mathbb{K}$
- We say vectors $\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \ldots, \bar{\mathbf{v}}_n$ in $\mathcal{V}$ are **linearly dependent** if there exist scalars $a_1, a_2, \ldots, a_n$, not all zero, such that

$$a_1\bar{\mathbf{v}}_1 + a_2\bar{\mathbf{v}}_2 + \cdots + a_n\bar{\mathbf{v}}_n = \mathbf{0}$$

- Otherwise the vectors are **linearly independent**

Linear independence (LI)

- In other words, vectors $\bar{\mathbf{v}}_1, \ \bar{\mathbf{v}}_2, \ . \ . \ . \ , \bar{\mathbf{v}}_n$ in $\mathcal{V}$ are **linearly independent**, if

$$a_1\bar{\mathbf{v}}_1 + a_2\bar{\mathbf{v}}_2 + \cdots + a_n\bar{\mathbf{v}}_n = \mathbf{0}$$
$$\Leftrightarrow a_1 = a_2 = \ldots = a_n = 0$$

Linear dependence

- $\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \ldots, \bar{\mathbf{v}}_n$ in $\mathcal{V}$ are linearly dependent if there **exist a non-zero solution** to the equation

$$\begin{bmatrix} | & | & | & | \\ \bar{\mathbf{v}}_1 & \bar{\mathbf{v}}_2 & \ldots & \bar{\mathbf{v}}_n \\ | & | & | & | \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = 0$$

$$\mathbf{V}\bar{\mathbf{a}} = \mathbf{0}$$

LI/LD – Properties

- **Property 1:** Suppose $\mathbf{0}$ is one of the vectors $\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \ldots, \bar{\mathbf{v}}_n$.
  - Then the vectors are **linearly dependent**
- This is easily seen as follows.
- Suppose $\bar{\mathbf{v}}_1 = \mathbf{0}$. Then

$$1\bar{\mathbf{v}}_1 + 0\bar{\mathbf{v}}_2 + \cdots + 0\bar{\mathbf{v}}_n = \mathbf{0}$$

LI/LD – Properties

- **Property 2:** Suppose $\bar{\mathbf{v}} \neq 0$.
- Then $\bar{\mathbf{v}}$ by itself is **linearly independent** since

$$k\bar{\mathbf{v}} = \mathbf{0} \Rightarrow k = 0$$

LI/LD – Properties

- **Property 3:** Suppose one of the vectors $\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \ldots, \bar{\mathbf{v}}_n$ is a scalar multiple of another.
  - Then the vectors are **linearly dependent**

- Without loss of generality, let $\bar{\mathbf{v}}_1 = k\bar{\mathbf{v}}_2$. Then

$$1\bar{\mathbf{v}}_1 - k\bar{\mathbf{v}}_2 + 0\bar{\mathbf{v}}_3 + \cdots + 0\bar{\mathbf{v}}_n = \mathbf{0}$$

LI/LD – Properties

- **Property 4:** Two vectors are **linearly independent** if and only if **neither is a multiple of the other**

LI/LD – Properties

- **Property 5:** Suppose a set S of vectors is **linearly independent**, then any subset of S is **linearly independent**.

LI/LD – Example

- Consider the vectors below

$$\bar{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \quad \bar{\mathbf{u}}_2 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \quad \bar{\mathbf{u}}_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

- We examine if they are linearly independent or dependent

LI/LD – Example

- Form now the equation

$$x\bar{\mathbf{u}}_1 + y\bar{\mathbf{u}}_2 + z\bar{\mathbf{u}}_3$$

$$= x\begin{bmatrix}1\\-1\\2\end{bmatrix} + y\begin{bmatrix}2\\-1\\3\end{bmatrix} + z\begin{bmatrix}1\\1\\4\end{bmatrix} = \mathbf{0}$$

LI/LD – Example

- The **equivalent system** of equations is

$$x + 2y + z = 0$$
$$-x - y + z = 0$$
$$2x + 3y + 4z = 0$$

LI/LD – Example

- This can be reduced to the system

$$x + 2y + z = 0$$
$$y + 2z = 0$$
$$4z = 0$$

- Back substitution yields

$$x = y = z = 0$$

  - Therefore, $\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \bar{\mathbf{u}}_3$ are **linearly independent**

Linear combination

- Vectors $\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \ldots, \bar{\mathbf{v}}_n$ are **linearly dependent** if and only if one of them is a linear combination of the others
- Suppose $\bar{\mathbf{v}}_i$ is a **linear combination** of $\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_{i-1}, \bar{\mathbf{v}}_{i+1}, \ldots, \bar{\mathbf{v}}_n$

$$\bar{\mathbf{v}}_i = a_1\bar{\mathbf{v}}_1 + \ldots + a_{i-1}\bar{\mathbf{v}}_{i-1} + a_{i+1}\bar{\mathbf{v}}_{i+1} + \ldots + a_n\bar{\mathbf{v}}_n$$

Linear combination

- It follows that

$$\bar{\mathbf{v}}_i - a_1\bar{\mathbf{v}}_1 - \cdots - a_{i-1}\bar{\mathbf{v}}_{i-1} - a_{i+1}\bar{\mathbf{v}}_{i+1} - \cdots - a_n\bar{\mathbf{v}}_n = \mathbf{0}$$

- Hence the vectors are **linearly dependent**

Linear combination

- On the other hand suppose the vectors are **linearly dependent**

$$a_1\bar{\mathbf{v}}_1 + a_2\bar{\mathbf{v}}_2 + \cdots + a_n\bar{\mathbf{v}}_n = \mathbf{0}$$

- Let $a_i \neq 0$. We can write

$$\bar{\mathbf{v}}_i = -\frac{a_1}{a_i}\bar{\mathbf{v}}_1 - \cdots - \frac{a_{i-1}}{a_i}\bar{\mathbf{v}}_{i-1} - \frac{a_{i+1}}{a_i}\bar{\mathbf{v}}_{i+1} - \cdots - \frac{a_n}{a_i}\bar{\mathbf{v}}_n$$

- This completes the proof

02

Basis and dimension

Bases, dimension and their properties



Basis

- A set $S = \{\bar{\mathbf{u}}_1, \ \bar{\mathbf{u}}_2, \ldots, \ \bar{\mathbf{u}}_m\}$ of vectors is a **basis** of $\mathcal{V}$ if it is **linearly independent** and spans $\mathcal{V}$

Basis and dimension

- **Theorem:** Let $\mathcal{V}$ be a vector space such that one basis has m elements and another basis has n elements.

- Then

$$m = n$$

Dimension

- Thus, all bases of $\mathcal{V}$ have the **same number of elements** n
  - This unique number n is termed the **dimension** of $\mathcal{V}$

$$m = n$$

- Alternatively, the vector space $\mathcal{V}$ is said to be n-dimensional

Basis – example

- Consider the vectors

$$\bar{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \bar{\mathbf{e}}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ \ldots, \ \bar{\mathbf{e}}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

- These vectors are **linearly independent**.

Basis – example

- Furthermore, any vector in $\mathbb{K}^n$ can be written as a **linear combination** of the above vectors

$$\bar{\mathbf{u}} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = u_1\bar{\mathbf{e}}_1 + u_2\bar{\mathbf{e}}_2 + \ldots + u_n\bar{\mathbf{e}}_n$$

Basis – example

- Thus, $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n$ form a basis for $\mathbb{R}^n$
- Furthermore, its dimension is n

Basis – polynomials

- Vector space $\mathcal{P}_n(t)$ of all polynomials of degree $\leq n$:
- The set S of $n+1$ polynomials is a basis of $\mathcal{P}_n(t)$

$$S = \{1,\ t,\ t^2,\ \ldots, t^n\}$$

Properties of bases

- Let $\mathcal{V}$ be a vector space of **finite dimension** n. Then:
- Property 1: Any $n+1$ or more vectors in $\mathcal{V}$ are **linearly dependent**.

Properties of bases

- **Property 2:** Any **linearly independent** set $S = \{\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \ldots, \bar{\mathbf{u}}_n\}$ with n elements is a basis of $\mathcal{V}$.
- **Property 3:** Any spanning set $T = \{\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \ldots, \bar{\mathbf{v}}_n\}$ of $\mathcal{V}$ with n elements is a basis of $\mathcal{V}$

Dimension and subspace

- Let $\mathcal{W}$ be a **subspace** of an n-dimensional vector space $\mathcal{V}$.
  - Then $\dim \mathcal{W} \leq n$.
- In particular, if $\dim \mathcal{W} = \quad n$, then $\mathcal{W} = \mathcal{V}$

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Rank of matrix

Row space, column space and the rank of a matrix



Row space

- Let $\mathbf{A}$ be any $m \times n$ matrix over a field $\mathbb{K}$.
- Rows of an $m \times n$ matrix $\mathbf{A}$ may be viewed as vectors in $\mathbb{K}^n$
- Row space of $\mathbf{A}$, written $\mathrm{rowsp}(\mathbf{A})$,
  - is the subspace of $\mathbb{K}^n$ spanned by the rows of $\mathbf{A}$

Rank of a matrix

- The rank of a matrix $\mathbf{A}$, written $\mathrm{rank}(\mathbf{A})$,
  - is equal to the **maximum number of linearly independent rows** of $\mathbf{A}$
  - or, equivalently, the **dimension of the row space** of $\mathbf{A}$

Column space

- Columns of an $m \times n$ matrix $\mathbf{A}$ may be viewed as vectors in $\mathbb{K}^m$
- The **column space** of $\mathbf{A}$, written $\mathrm{colsp}(\mathbf{A})$,
  - is the subspace of $\mathbb{K}^m$ **spanned by the columns** of $\mathbf{A}$

Dimension of Row/Col Space

- THEOREM: The maximum number of **linearly independent rows** of any matrix $\mathbf{A}$

  - is **equal** to the **maximum number of linearly independent columns** of $\mathbf{A}$.

Rank of a matrix

- Thus, the **dimension of the row space** of $\mathbf{A}$ is **equal to the dimension of the column space** of $\mathbf{A}$.

- Therefore,

$$\begin{aligned}\dim \mathbf{rowsp}(\mathbf{A}) &= \dim \mathbf{colsp}(\mathbf{A}) \\ &= \operatorname{rank}(\mathbf{A})\end{aligned}$$

Rank of a matrix – Ex.

- Consider the matrix $\mathbf{A}$ below and its **echelon** form $\mathbf{B}$

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 & 2 & 0 & 1 \\ 2 & -1 & 5 & 3 & 2 & 5 \\ 3 & -1 & 8 & 5 & 5 & 7 \\ 1 & 2 & 5 & 1 & 8 & 6 \\ 2 & -1 & 5 & 4 & 3 & 3 \end{bmatrix} \sim \mathbf{B} = \begin{bmatrix} 1 & -1 & 2 & 2 & 0 & 1 \\ 0 & 1 & 1 & -1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of a matrix – Ex.

- $\mathbf{A}$ and $\mathbf{B}$ are **row equivalent**, so they have the **same row space**
- Further, considering the number of pivots

$$\text{rank}\,(\mathbf{A}) = \text{rank}\,(\mathbf{B}) = 3$$

Row space – Example

- Moreover, $\mathbf{B}$ is in **echelon form**,
- So its nonzero rows are linearly independent
  - and hence form a **basis of the row space** of $\mathbf{B}$.
- Since $\mathbf{A}$ and $\mathbf{B}$ are row equivalent, they also form a basis for row space of $\mathbf{A}$.

Row space – Example

$$\begin{bmatrix} 1 & -1 & 2 & 2 & 0 & 1 \\ 0 & 1 & 1 & -1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row space – Example

- Basis of the row space of $\mathbf{A}$ and $\mathbf{B}$ is

$$\begin{bmatrix} 1 & -1 & 2 & 2 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 & -1 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & -2 \end{bmatrix}$$

Column space – Example

$$\mathbf{B} = \begin{bmatrix} \mathbf{1} & \mathbf{-1} & 2 & \mathbf{2} & 0 & 1 \\ \mathbf{0} & \mathbf{1} & 1 & \mathbf{-1} & 2 & 3 \\ \mathbf{0} & \mathbf{0} & 0 & \mathbf{1} & 1 & -2 \\ \mathbf{0} & \mathbf{0} & 0 & \mathbf{0} & 0 & 0 \\ \mathbf{0} & \mathbf{0} & 0 & \mathbf{0} & 0 & 0 \end{bmatrix}$$

- Columns $\bar{\mathbf{c}}_1$, $\bar{\mathbf{c}}_2$, $\bar{\mathbf{c}}_4$ of $\mathbf{B}$ associated with pivots are **linearly independent**

Row space – Example

$$\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-1} & 2 & \mathbf{2} & 0 & 1 \\ \mathbf{2} & \mathbf{-1} & 5 & \mathbf{3} & 2 & 5 \\ \mathbf{3} & \mathbf{-1} & 8 & \mathbf{5} & 5 & 7 \\ \mathbf{1} & \mathbf{2} & 5 & \mathbf{1} & 8 & 6 \\ \mathbf{2} & \mathbf{-1} & 5 & \mathbf{4} & 3 & 3 \end{bmatrix}$$

- Therefore, $\bar{\mathbf{c}}_1$, $\bar{\mathbf{c}}_2$, $\bar{\mathbf{c}}_4$ of $\mathbf{A}$ are also **linearly independent**
  - And form a basis for $\mathrm{colsp}(\mathbf{A})$

Row space – Example

- Thus, the **basis of the column space** of $\mathbf{A}$ is given as

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ -1 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 3 \\ 5 \\ 1 \\ 4 \end{bmatrix}$$

04

Null space

Solutions of Ax = 0 and the rank–nullity result



Homogeneous system

- Consider again a **homogeneous system** $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$ of linear equations over $\mathbb{K}$ with n unknowns.
- As seen earlier, the solution set $\mathcal{W}$ of such a system is a **subspace** of $\mathbb{K}^n$.
  - This is termed the **null space** of $\mathbf{A}$.

Dimension of Null space

- **THEOREM:** The dimension of the solution space $\mathbf{W}$ of a **homogeneous system** $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$ is $n - r$,
  - where n is the **number of unknowns** and r is the **rank of the coefficient matrix** $\mathbf{A}$.

Nullity

- The **dimension of the null space** of a matrix is also termed its **nullity**.

- Therefore, we have the result

$$\begin{aligned}\text{Rank} + \text{nullity} &= r + (n - r) = n \\ &= \text{number of columns of } \mathbf{A}\end{aligned}$$

Null space basis-Example

- Since **null space** of a matrix is a vector space one can find a basis for it

- Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

Null space basis-Example

- Basis for the null space can be found as follows

- Set

$$\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$$

$$\Rightarrow \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{0}$$

Null space basis-Example

- Solving the equations yields

$$x_1 - 3x_2 + 0x_3 + 2x_4 = 0$$
$$\Rightarrow x_1 = 3x_2 - 2x_4$$
$$x_3 - 4x_4 = 0$$
$$\Rightarrow x_3 = 4x_4$$

Null space basis-Example

- Therefore, any vector of the **null space** can be characterized as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_2 - 2x_4 \\ x_2 \\ 4x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

Null space basis-Example

- Therefore, **basis of the null space** is

$$\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

05

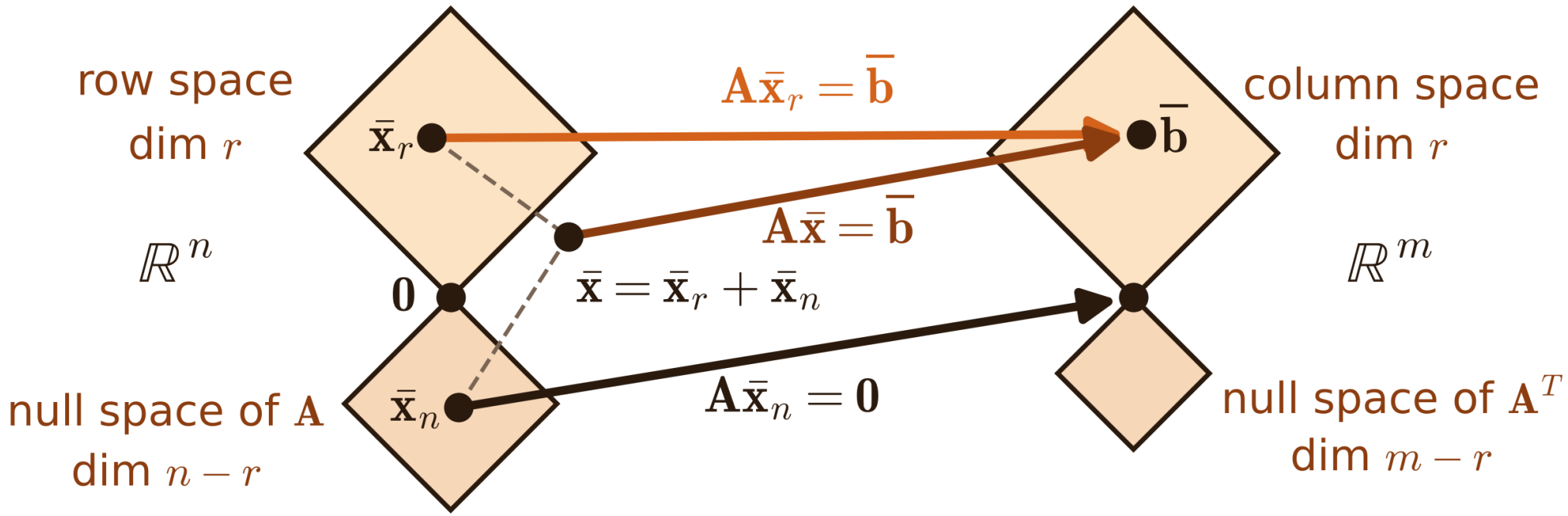
Four fundamental subspaces

Row, column and null spaces of A and its transpose



Four fundamental subspaces

- There are **four fundamental subspaces** of any matrix



Four fundamental subspaces

- These four fundamental subspaces of matrix $\mathbf{A}$ of size $m \times n$ and rank r are
  - **Row space of $\mathbf{A}$ of dim r**
  - **Column space of $\mathbf{A}$ of dim r**
  - **Null space of $\mathbf{A}$ of dim $n - r$**
  - **Null space of $\mathbf{A}^T$ of dim $m - r$**

Four fundamental subspaces

- These four fundamental subspaces satisfy the properties
  - Row space of $\mathbf{A}$ is **orthogonal** to Null space of $\mathbf{A}$
  - Column space of $\mathbf{A}$ is **orthogonal** to Null space of $\mathbf{A}^T$

Summary

- LI: $a_1\bar{\mathbf{v}}_1 + \cdots + a_n\bar{\mathbf{v}}_n = \mathbf{0}$ only when every $a_i = 0$
- Vectors are dependent if and only if one is a combination of the others
- A basis is linearly independent and spans V; all bases have dim V elements
- $\text{rank}\,(\mathbf{A}) = \dim \text{rowsp}\,(\mathbf{A}) = \dim \text{colsp}\,(\mathbf{A})$
- Nullity is $n - r$, so rank + nullity = number of columns
- Four fundamental subspaces: row, column, null space of A and of its transpose



Thank You!

Questions?