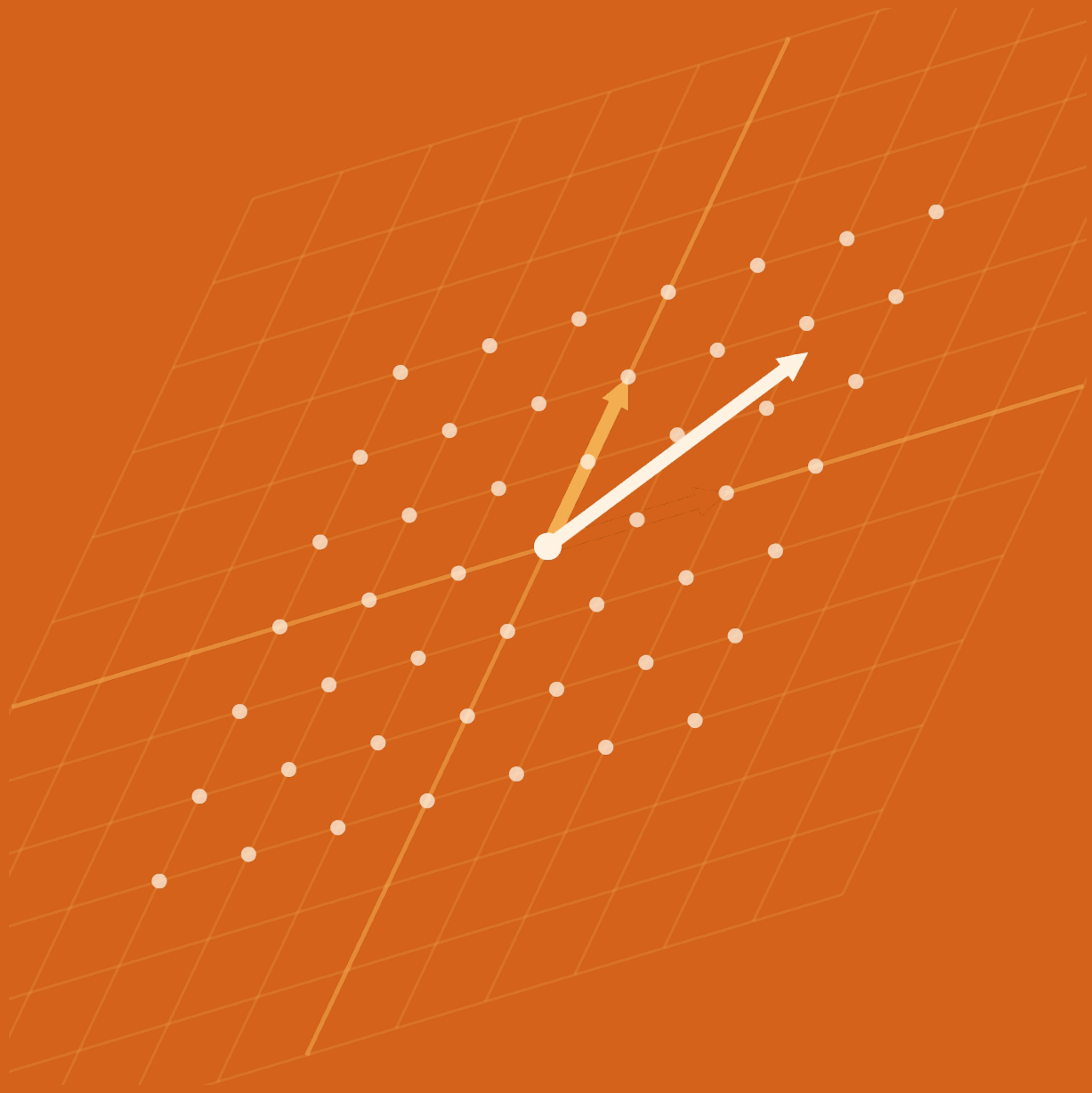
LINEAR ALGEBRA FOR AIML AND 6G

# Vector spaces: Subspaces and spans



Lecture 11

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# Outline

1. **Subspace**
   Subsets that are vector spaces in their own right

2. **Linear spans**
   All linear combinations of a set of vectors

3. **Row/ column spaces**
   Row space, column space and row equivalence

# 01

# Subspace

Subsets that are vector spaces in their own right



# Subspace

- Let $\mathcal{V}$ be a vector space over a field $\mathbb{K}$ and let $\mathcal{W}$ be a subset of $\mathcal{V}$.
- Then $\mathcal{W}$ is a **subspace** of $\mathcal{V}$ if $\mathcal{W}$ is itself a vector space over $\mathbb{K}$
  - with respect to the operations of vector addition and **scalar multiplication** on $\mathcal{V}$

# Subspace

- Suppose $\mathcal{W}$ is a subset of a vector space $\mathcal{V}$.
- Then $\mathcal{W}$ is a subspace of $\mathcal{V}$ if the following **two conditions** hold:
  - The zero vector **0** belongs to $\mathcal{W}$
  - For every $\bar{\mathbf{u}}, \ \bar{\mathbf{v}} \in \mathcal{W}$; $a, \ b \in \mathbb{K}$, the linear combination $a\bar{\mathbf{u}} + \ \ b\bar{\mathbf{v}} \in \mathcal{W}$.

# Trivial subspaces

- Now let $\mathcal{V}$ be any vector space.
- Then $\mathcal{V}$ automatically contains two subspaces:
  - the set $\{\mathbf{0}\}$ consisting of the zero vector alone
  - and the **whole space** $\mathcal{V}$ itself.
- These are sometimes called the **trivial subspaces** of $\mathcal{V}$.

# Subspace example

- Consider the vector space $\mathcal{V} = \quad \mathbb{R}^3$.
- Let $\mathcal{U}$ consist of all vectors in $\mathbb{R}^3$ whose entries are equal; that is,

$$\mathcal{U} = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, a = b = c \right\}$$

- Note that $\mathbf{0} \in \mathcal{U}$

# Subspace example

- For $\bar{\mathbf{u}} = [a\ a\ a]^T$ and $\bar{\mathbf{v}} = [b\ b\ b]^T$ in $U$

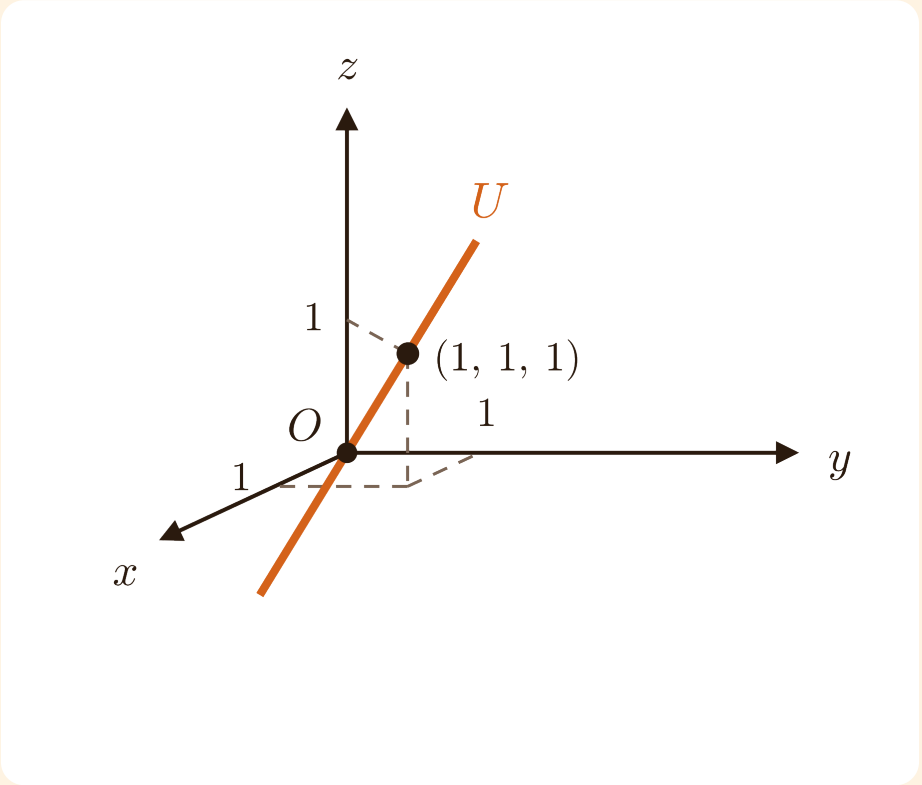
- Then

$$\alpha\bar{\mathbf{u}} + \beta\bar{\mathbf{v}} = \begin{bmatrix} \alpha a + \beta b \\ \alpha a + \beta b \\ \alpha a + \beta b \end{bmatrix} \in \mathcal{U}$$

- Thus, $\mathcal{U}$ is a **subspace** of $\mathbb{R}^3$.

# Subspace example

- Geometrically, $\mathcal{U}$ is the line **through the origin** $\mathbf{O}$ and the point $[1,\ 1,\ 1]^T$ as shown



# Subspace matrices

- Let $V = \mathbb{R}^{n \times n}$ the vector space of $n \times n$ matrices.
- Let $\mathcal{W}_1$ be the subset of all **(upper) triangular matrices**
  - and let $\mathcal{W}_2$ be the subset of all **symmetric matrices**.

# Subspace matrices

- Then $\mathcal{W}_1$ is a subspace of $\mathcal{V}$,
  - because $\mathcal{W}_1$ contains the zero matrix 0 and
  - $\mathcal{W}_1$ is closed under **matrix addition** and **scalar multiplication**;
- Hence, the sum and scalar multiple of such triangular matrices are also **triangular**.

# Subspace matrices

- Similarly, $\mathcal{W}_2$ is also a **subspace** of $\mathcal{V}$

# Subspace polynomials

- Let $\mathcal{V} = \mathcal{P}(t)$, the vector space of polynomials.
- Let $\mathcal{Q}(t)$ be the collection of polynomials with only **even powers** of $t$.

# Subspace polynomials

- For example, the following are polynomials in $\mathcal{Q}(t)$

$$p_1(t) = 2 - 6t^2 + 8t^4$$

$$p_2(t) = -5 + t^4 + 4t^8 - 2t^{10}$$

- Then $\mathcal{Q}(t)$ is a **subspace** of $\mathcal{P}(t)$

# Intersection of subspaces

- Let $\mathcal{U}$ and $\mathcal{W}$ be subspaces of a vector space $\mathcal{V}$.
  - We show that the **intersection** $\mathcal{U} \cap \mathcal{W}$ is also a subspace of $\mathcal{V}$.
- Clearly, $\mathbf{0} \in \mathcal{U}$ and $\mathbf{0} \in \mathcal{W}$, because $\mathcal{U}$ and $\mathcal{W}$ are subspaces; whence $\mathbf{0} \in \mathcal{U} \cap \mathcal{W}$

# Intersection of subspaces

- Suppose $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ belong to the intersection $\mathcal{U} \cap \mathcal{W}$.
- Then $\bar{\mathbf{u}}, \ \bar{\mathbf{v}} \in \mathcal{U}$ and $\bar{\mathbf{u}}, \ \bar{\mathbf{v}} \in \mathcal{W}$.

# Intersection of subspaces

- Further, because $\mathcal{U}$ and $\mathcal{W}$ are subspaces, for any scalars $a, b \in \mathbb{K}$

$$a\bar{\mathbf{u}} + b\bar{\mathbf{v}} \in \mathcal{U}, a\bar{\mathbf{u}} + b\bar{\mathbf{v}} \in \mathcal{W}$$

$$\Rightarrow a\bar{\mathbf{u}} + b\bar{\mathbf{v}} \in \mathcal{U} \cap \mathcal{W}$$

- $\Rightarrow \mathcal{U} \cap \mathcal{W}$ is a **subspace**

# Homogeneous system

- Let $\mathbf{A}\bar{\mathbf{x}} = \bar{\mathbf{b}}$ be a system of linear equations in $n$ unknowns.
- Suppose the system has the form

$$\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$$

- Let $\mathcal{W}$ be its solution set.
- Because $\mathbf{A0} = \mathbf{0}$, the zero vector $\mathbf{0} \in \mathcal{W}$.

# Homogeneous system

- Moreover, suppose $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ belong to $\mathcal{W}$.
- Then $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ are solutions of $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$,
- or, in other words

$$\mathbf{A}\bar{\mathbf{u}} = \mathbf{0}, \mathbf{A}\bar{\mathbf{v}} = \mathbf{0}$$

# Homogeneous system

- Therefore, for any scalars $a$ and $b$, we have

$$\begin{aligned}\mathbf{A}\left(a\bar{\mathbf{u}} + b\bar{\mathbf{v}}\right) &= a\mathbf{A}\bar{\mathbf{u}} + b\mathbf{A}\bar{\mathbf{v}} \\ &= a\mathbf{0} + b\mathbf{0} = \mathbf{0} + \mathbf{0} = \mathbf{0}\end{aligned}$$

# Homogeneous system

- Thus, $a\bar{\mathbf{u}} + \quad b\bar{\mathbf{v}}$ belongs to $\mathcal{W}$,
  - Accordingly, $\mathcal{W}$ is a subspace of $\mathbb{K}^n$
- This is termed the **null space** of $\mathbf{A}$

# 02

# Linear spans

All linear combinations of a set of vectors



# Linear span

- Suppose $\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \ldots, \bar{\mathbf{u}}_m$ are any vectors in a vector space $\mathcal{V}$.
- Recall that any vector of the form

$$a_1\bar{\mathbf{u}}_1 + a_2\bar{\mathbf{u}}_2 + \ldots + a_m\bar{\mathbf{u}}_m$$

  - where the $a_i$ are scalars, is called a **linear combination** of $\bar{\mathbf{u}}_i$

# Linear span

- The collection of all such **linear combinations**, denoted by

$$\text{span}\left(\bar{\mathbf{u}}_1,\ \bar{\mathbf{u}}_2,\ldots,\ \bar{\mathbf{u}}_m\right),\ \ \text{span}(\bar{\mathbf{u}}_i)$$

# Linear span – properties

- Clearly the zero vector $\mathbf{0}$ belongs to $\mathrm{span}(\bar{\mathbf{u}}_i)$, because

$$\mathbf{0} = 0\bar{\mathbf{u}}_1 + 0\bar{\mathbf{u}}_2 + \ldots + 0\bar{\mathbf{u}}_m$$

- Furthermore, suppose $\bar{\mathbf{v}}$ and $\bar{\mathbf{w}}$ belong to $\mathrm{span}(\bar{\mathbf{u}}_i)$, say,

$$\bar{\mathbf{v}} = a_1\bar{\mathbf{u}}_1 + a_2\bar{\mathbf{u}}_2 + \ldots + a_m\bar{\mathbf{u}}_m$$
$$\bar{\mathbf{w}} = b_1\bar{\mathbf{u}}_1 + b_2\bar{\mathbf{u}}_2 + \ldots + b_m\bar{\mathbf{u}}_m$$

# Linear span – properties

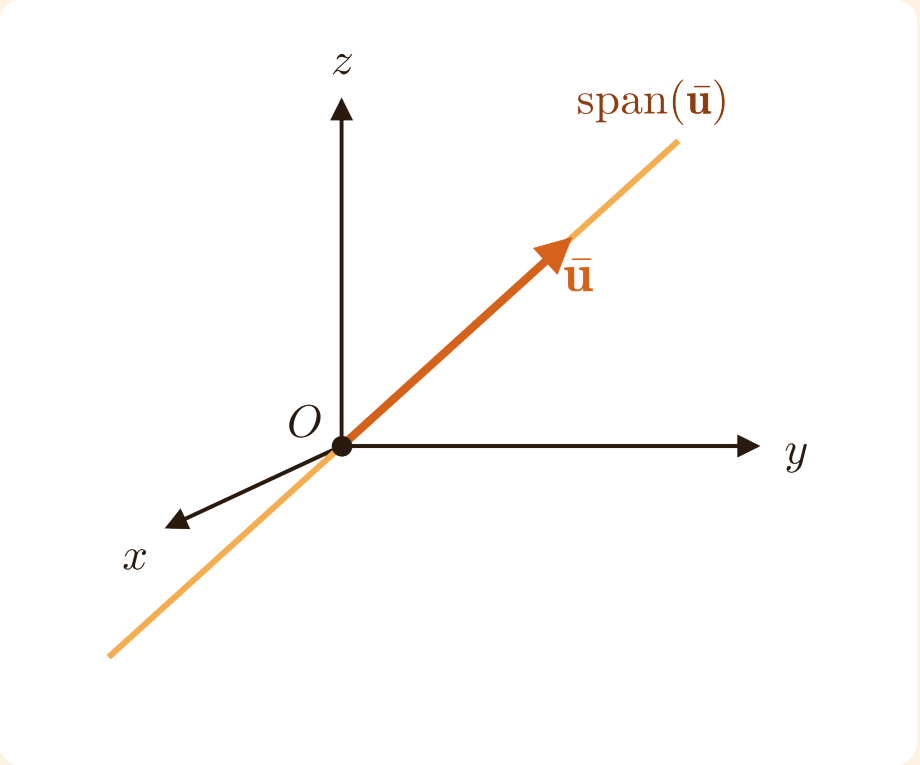
- Then

$$\alpha\bar{\mathbf{v}} + \beta\bar{\mathbf{w}} = (\alpha a_1 + \beta b_1)\bar{\mathbf{u}}_1 + \cdots + (\alpha a_m + \beta b_m)\bar{\mathbf{u}}_m \in \text{span}(\bar{\mathbf{u}}_i)$$

- Thus, $\text{span}(\bar{\mathbf{u}}_i)$ is a **subspace** of $\mathcal{V}$

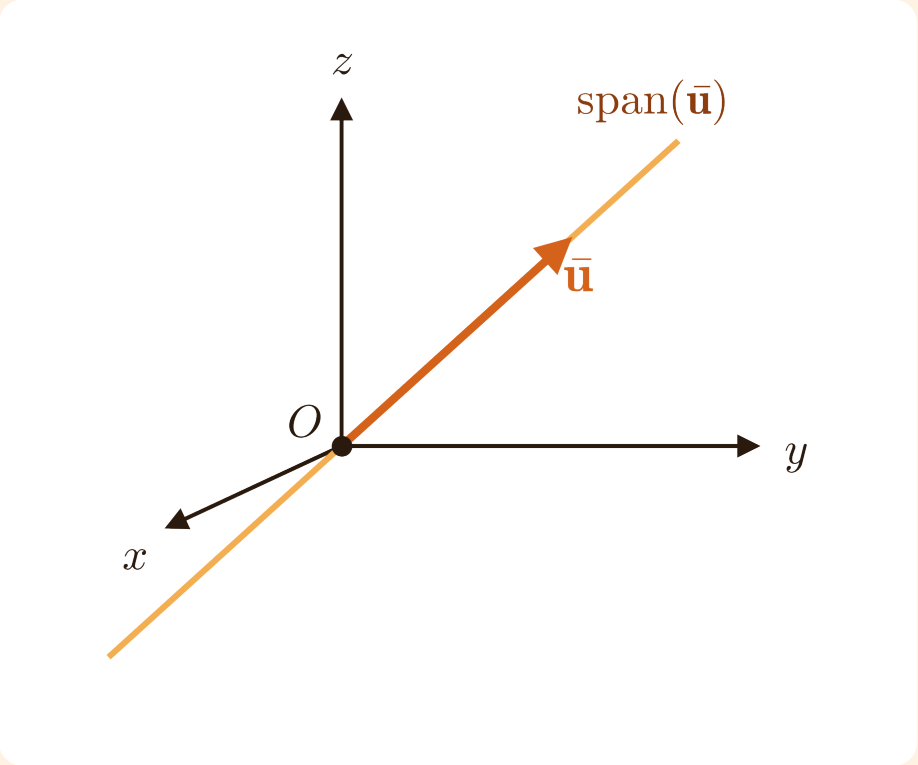
# Linear span – Line

- Let $\bar{\mathbf{u}}$ be any nonzero vector in $\mathbb{R}^3$. Then $\text{span}(\bar{\mathbf{u}})$ consists of all **scalar multiples** of $\bar{\mathbf{u}}$.



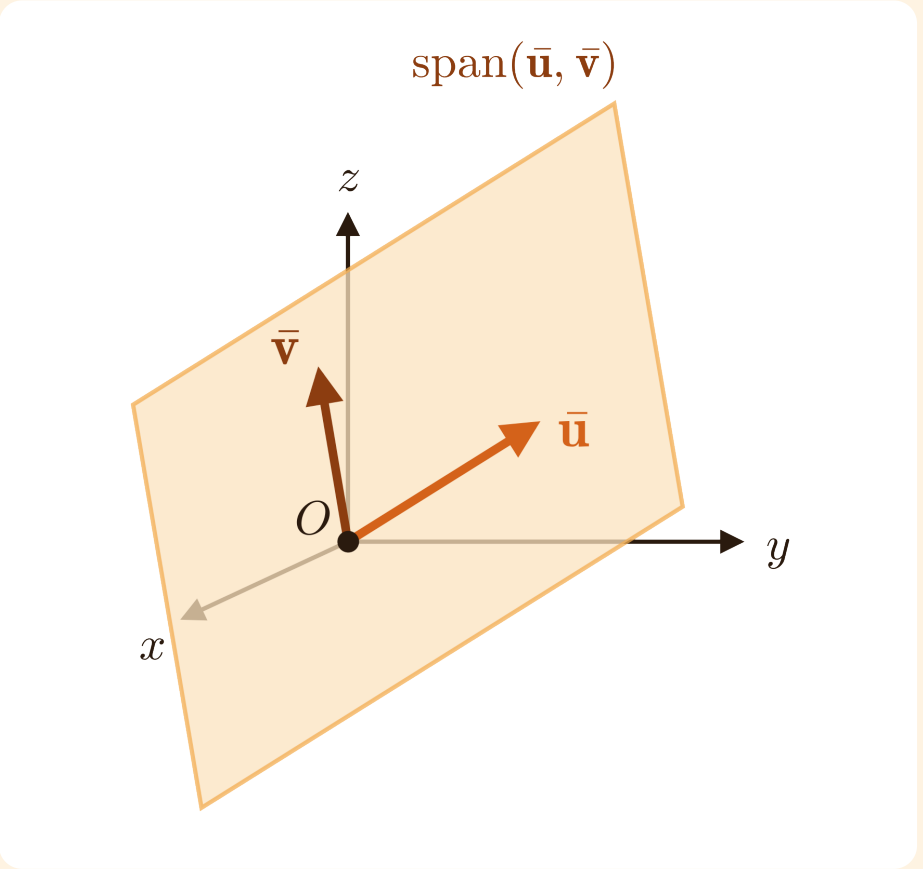
# Linear span – Line

- Geometrically, $\mathbf{span}(\bar{\mathbf{u}})$ is the **line through the origin** $\mathcal{O}$ and the endpoint of $\bar{\mathbf{u}}$



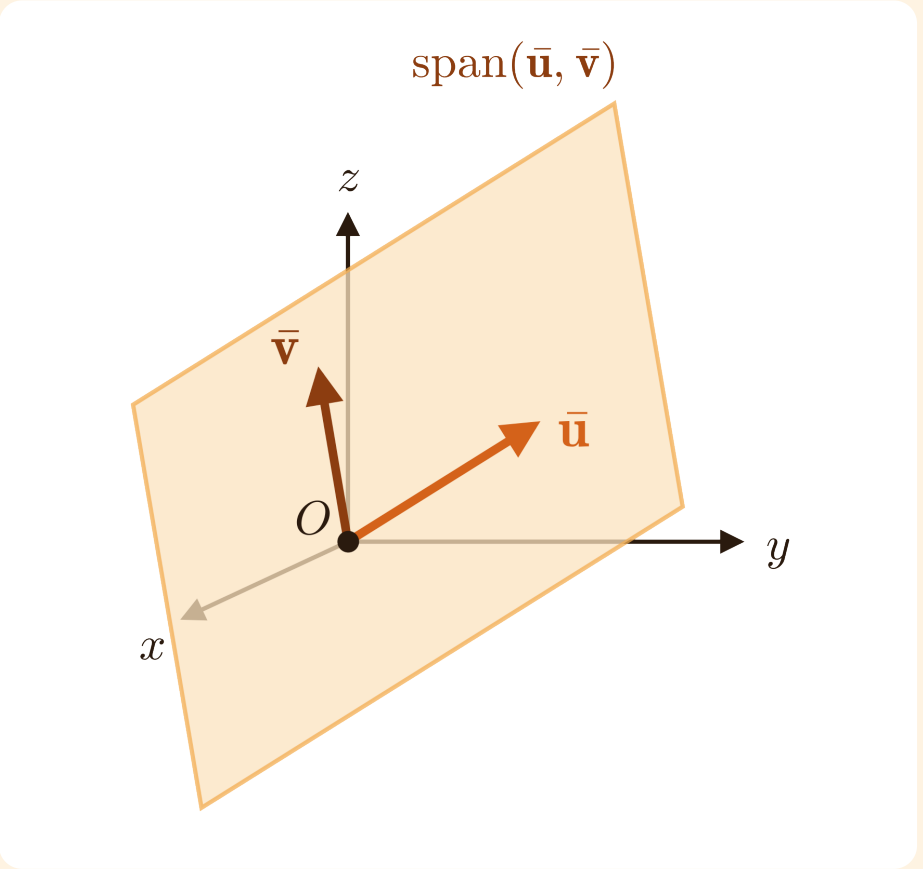
# Linear span – Plane

- Let $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ be vectors in $\mathbb{R}^3$ that are not multiples of each other.



# Linear span – Plane

- Then $\mathrm{span}(\bar{\mathbf{u}},\bar{\mathbf{v}})$ is the **plane through the origin** $\mathcal{O}$ and the endpoints of $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$



# Linear span – properties

- Consider the vectors

$$\bar{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{\mathbf{e}}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \bar{\mathbf{e}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Then

$$\text{span}\left(\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3\right) = \mathbb{R}^3$$

# 03

# Row/ column spaces

Row space, column space and row equivalence



# Column space

- Let $\mathbf{A} = [a_{ij}]$ be an arbitrary $m \times n$ matrix with columns $\bar{\mathbf{a}}_1$, $\bar{\mathbf{a}}_2$,…, $\bar{\mathbf{a}}_n$.

- Then, **column space** of $\mathbf{A}$ is

$$\text{span}\left(\bar{\mathbf{a}}_1, \bar{\mathbf{a}}_2, \ldots, \bar{\mathbf{a}}_n\right)$$

# Row space

- Similarly, one can define the **row space** of $\mathbf{A}$
- Note

$$\text{rowsp}\left(\mathbf{A}\right) = \text{colsp}\left(\mathbf{A}^T\right)$$

# Row operations

- Recall that matrices $\mathbf{A}$ and $\mathbf{B}$ are row equivalent, written $\mathbf{A} \sim \mathbf{B}$,
  - if $\mathbf{B}$ can be obtained from $\mathbf{A}$ by a sequence of elementary row operations.

$$\bar{\mathbf{r}}_i \leftrightarrow \bar{\mathbf{r}}_j, \quad k\bar{\mathbf{r}}_i \rightarrow \bar{\mathbf{r}}_i, \quad \bar{\mathbf{r}}_j + k\bar{\mathbf{r}}_i \rightarrow \bar{\mathbf{r}}_j$$

# Row operations

- Suppose $\mathbf{M}$ is the matrix obtained by applying one of the **elementary row operations**
- Then each row of $\mathbf{M}$ is a row of $\mathbf{A}$ or a **linear combination** of rows of $\mathbf{A}$.
  - Hence, the row space of $\mathbf{M}$ is contained in the **row space** of $\mathbf{A}$.

# Row operations

- On the other hand, we can apply the **inverse elementary row operation** on $\mathbf{M}$ to obtain $\mathbf{A}$;

- Hence, the **row space** of $\mathbf{A}$ is contained in the **row space** of $\mathbf{M}$.

# Equivalent row spaces

- Accordingly, $\mathbf{A}$ and $\mathbf{M}$ have the same row space.
- THEOREM: **Row equivalent matrices** have the same row space.

# Equivalent row spaces

- Theorem: Suppose $\mathbf{A}$ and $\mathbf{B}$ are **row canonical matrices**.
- Then $\mathbf{A}$ and $\mathbf{B}$ have the same **row space**
  - if and only if they have the **same nonzero rows**.

# Equivalent row space–Ex.

- Consider the following two sets of vectors $\bar{\mathbf{u}}_i$ and $\bar{\mathbf{w}}_i$

$$\bar{\mathbf{u}}_1^T = \begin{bmatrix} 1 & -3 & 2 & 12 \end{bmatrix},$$

$$\bar{\mathbf{u}}_2^T = \begin{bmatrix} 2 & -6 & -1 & -1 \end{bmatrix},$$

$$\bar{\mathbf{u}}_3^T = \begin{bmatrix} 3 & -9 & 1 & 11 \end{bmatrix}$$

$$\bar{\mathbf{w}}_1^T = \begin{bmatrix} 1 & -3 & -1 & -3 \end{bmatrix},$$

$$\bar{\mathbf{w}}_2^T = \begin{bmatrix} 2 & -6 & 3 & 19 \end{bmatrix}$$

# Equivalent row space–Ex.

- Let $\mathcal{U} = \text{span}(\bar{\mathbf{u}}_\mathbf{i})$, $\mathcal{W} = \text{span}(\bar{\mathbf{w}}_i)$
- Form matrix $\mathbf{A}$ with rows $\bar{\mathbf{u}}_i^T$

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 2 & 12 \\ 2 & -6 & -1 & -1 \\ 3 & -9 & 1 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 12 \\ 0 & 0 & -5 & -25 \\ 0 & 0 & -5 & -25 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

# Equivalent row space–Ex.

- Form matrix $\mathbf{B}$ with rows $\bar{\mathbf{w}}_i^T$

$$\mathbf{B} = \begin{bmatrix} 1 & -3 & -1 & -3 \\ 2 & -6 & 3 & 19 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -1 & -3 \\ 0 & 0 & 5 & 25 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

# Equivalent row space–Ex.

- Thus, **row canonical forms** of $\mathbf{A}$ and $\mathbf{B}$ have the same non-zero rows

$$\begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

- The row spaces of $\mathbf{A}$ and $\mathbf{B}$ are identical.
- Therefore,

$$\mathcal{U} = \text{span}\left(\bar{\mathbf{u}}_i\right) = \mathcal{W} = \text{span}\left(\bar{\mathbf{w}}_i\right)$$

# Summary

- A subspace contains **0** and is closed under linear combinations
- Examples: lines through $O$, triangular or symmetric matrices, even polynomials
- Intersection of subspaces is a subspace; so is the solution set of $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$
- $\mathrm{span}(\bar{\mathbf{u}}_i)$, all linear combinations, is a subspace
- Row-equivalent matrices have the same row space; $\mathrm{rowsp}(\mathbf{A}) = \mathrm{colsp}(\mathbf{A}^T)$



# Thank You!

Questions?