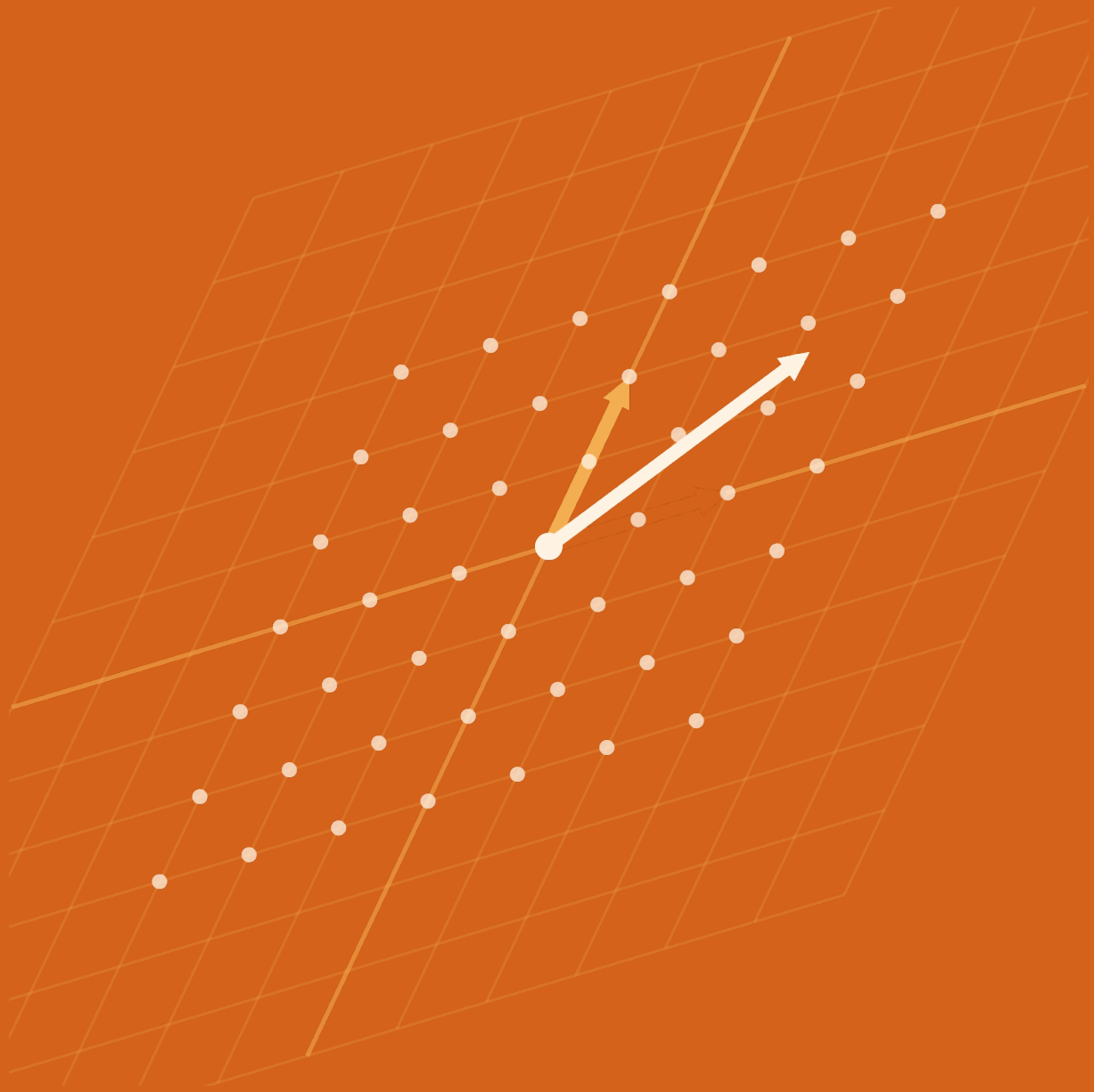
LINEAR ALGEBRA FOR AIML AND 6G

Vector spaces: Definition and Examples



Lecture 10

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Outline

1. **Vector spaces**
   Axioms, and the spaces K^n, polynomials, matrices and functions

2. **Linear combination**
   Writing a vector as a combination of given vectors

3. **Spanning sets**
   Sets of vectors whose combinations give the whole space

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Vector spaces

Axioms, and the spaces K^n, polynomials, matrices and functions



Vector space

- A vector space $\mathcal{V}$, comprises elements $\bar{\mathbf{u}}$, $\bar{\mathbf{v}}$, $\bar{\mathbf{w}}$ termed **vectors**,
- over an **arbitrary field** $\mathbb{K}$,
  - whose elements are termed scalars $a, b, c \in \mathbb{K}$
- $\mathbb{K}$ can be $\mathbb{R}$ or $\mathbb{C}$

Vector space

- Let $\mathcal{V}$ be a nonempty set with two operations,
- **Vector Addition**: This assigns to any $\bar{\mathbf{u}}, \bar{\mathbf{v}} \in V$ a sum $\bar{\mathbf{u}} + \bar{\mathbf{v}} \in \mathcal{V}$
- **Scalar Multiplication**: This assigns to any $\bar{\mathbf{u}} \in \mathcal{V}$, $k \in \mathbb{K}$, a product $k\bar{\mathbf{u}} \in \mathcal{V}$.

Vector space

- $\mathcal{V}$ is termed a **vector space** (over the field $\mathbb{K}$) if the following axioms hold for any vectors $\bar{\mathbf{u}},\ \bar{\mathbf{v}},\ \bar{\mathbf{w}} \in \mathcal{V}$

$$(\bar{\mathbf{u}} + \bar{\mathbf{v}}) + \bar{\mathbf{w}} = \bar{\mathbf{u}} + (\bar{\mathbf{v}} + \bar{\mathbf{w}})$$

- There is a vector in $\mathcal{V}$, denoted by $\mathbf{0}$ and termed the **zero vector**, such that for any $\bar{\mathbf{u}} \in \mathcal{V}$

$$\bar{\mathbf{u}} + \mathbf{0} = \mathbf{0} + \bar{\mathbf{u}} = \bar{\mathbf{u}}$$

Vector space

- For each $\bar{\mathbf{u}} \in \mathcal{V}$, there is a vector in $\mathcal{V}$, denoted by $-\bar{\mathbf{u}}$, called the **negative of** $\bar{\mathbf{u}}$, such that

$$\bar{\mathbf{u}} + (-\bar{\mathbf{u}}) = (-\bar{\mathbf{u}}) + \bar{\mathbf{u}} = \mathbf{0}$$

$$\bar{\mathbf{u}} + \bar{\mathbf{v}} = \bar{\mathbf{v}} + \bar{\mathbf{u}}$$

$$k(\bar{\mathbf{u}} + \bar{\mathbf{v}}) = k\bar{\mathbf{u}} + k\bar{\mathbf{v}}, k \in \mathbb{K}$$

Vector space

- For scalars $a, b \in \mathbb{K}$ and $\bar{\mathbf{u}} \in \mathcal{V}$

$$(a+b)\,\bar{\mathbf{u}} = a\bar{\mathbf{u}} + b\bar{\mathbf{u}}$$

$$(ab)\,\bar{\mathbf{u}} = a\,(b\bar{\mathbf{u}})$$

$$1\bar{\mathbf{u}} = \bar{\mathbf{u}}$$

Vector space

- For any scalar $k \in \mathbb{K}$ and $\mathbf{0} \in \mathcal{V}$

$$k\mathbf{0} = \mathbf{0}$$

- For $0 \in \mathbb{K}$ and any vector $\bar{\mathbf{u}} \in \mathcal{V}$

$$0\bar{\mathbf{u}} = \mathbf{0}$$

Vector space

- If $k\bar{\mathbf{u}} = \mathbf{0}$, where $k \in \mathbb{K}$ and $\bar{\mathbf{u}} \in \mathcal{V}$, then
- $k = 0$ **or** $\bar{\mathbf{u}} = \mathbf{0}$
- For any $k \in \mathbb{K}$ and any $\bar{\mathbf{u}} \in \mathcal{V}$

$$(-k)\,\bar{\mathbf{u}} = k\,(-\bar{\mathbf{u}}) = -k\bar{\mathbf{u}}$$

Space

- Let $\mathbb{K}$ be an arbitrary field
- The notation $\mathbb{K}^n$ is used to denote the set of all n-**tuples** of elements in $\mathbb{K}$
- Let $a_1, a_2, \ldots, a_n \in \mathbb{K}$

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{K}^n$$

Space – Vector addition

- $\mathbb{K}^n$ is a vector space over $\mathbb{K}$ using the following operations

- **Vector addition**

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

Space – Scalar mult.

- **Scalar multiplication**

$$k \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ka_1 \\ ka_2 \\ \vdots \\ ka_n \end{bmatrix}$$

Space – Negative

- The **negative of a vector** is defined by

$$-\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} -a_1 \\ -a_2 \\ \vdots \\ -a_n \end{bmatrix}$$

Space – Zero vector

- The zero vector in $\mathbb{K}^n$ is the n-tuple of zeros

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Space

- When $\mathbb{K} = \mathbb{R}$, the space is $\mathbb{R}^n$
- $\mathbb{R}^n$ is a **vector space** over $\mathbb{R}$
- $\mathbb{C}^n$ is a **vector space** over $\mathbb{C}$

Polynomial space

- Let $\mathcal{P}(t)$ denote the set of **all polynomials** of the form

$$p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_s t^s$$

Polynomial space

- Let $\mathcal{P}_n(t)$ denote the set of all polynomials of the form

$$p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_s t^s$$

- where $s \leq n$ i.e. degree $\leq n$

Polynomial operations

- Then $\mathcal{P}(t)$ and $\mathcal{P}_n(t)$ are vector spaces over $\mathbb{K}$ using the following operations:

- **Vector Addition:** Here $p(t) + q(t)$ is the usual operation of **addition of polynomials**.

Polynomial operations

- **Scalar Multiplication**: Here $kp(t)$ is the usual operation of the product of a scalar k and a polynomial $p(t)$.

- The **zero polynomial** 0 is the zero vector in $\mathcal{P}_n(t)$ or $\mathcal{P}(t)$.

Matrix space

- The notation $\mathbb{K}^{m\times n}$ will be used to denote the set of all $m \times n$ matrices with entries from $\mathbb{K}$.

- $\mathbb{K}^{m\times n}$ is a **vector space** over $\mathbb{K}$ with respect to the usual operations of **matrix addition** and **scalar multiplication of matrices**

Function space

- Let $\mathcal{X}$ be a **non-empty set** and let $\mathbb{K}$ be an arbitrary field
- Let $\mathfrak{F}(\mathcal{X})$ denote the **set of all functions** of $\mathcal{X}$ into $\mathbb{K}$

Function space

- Then $\mathfrak{F}(\mathcal{X})$ is a **vector space** over $\mathbb{K}$
- The sum of two functions f and g in $\mathfrak{F}(\mathcal{X})$ is defined as

$$(f+g)(x) = f(x) + g(x)$$

- The product of a scalar $k \in \mathbb{K}$ and $f \in \mathfrak{F}(\mathcal{X})$ is defined as

$$(kf)(x) = kf(x)$$

Function space

- The **zero function** maps every element $x \in \mathcal{X}$ to $0 \in \mathbb{K}$

$$0\,(x) = 0$$

- Also for any function, the **negative of the function** is defined by

$$(-f)\,(x) = -f\,(x)$$

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Linear combination

Writing a vector as a combination of given vectors



Linear combination

- Let $\mathcal{V}$ be a vector space over a field $\mathbb{K}$
- A vector $\bar{\mathbf{v}} \in \mathcal{V}$ is a **linear combination** of $\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \ldots, \bar{\mathbf{u}}_m$ if there exists scalars $a_1, a_2, \ldots, a_m$ in $\mathbb{K}$ such that

$$\bar{\mathbf{v}} = a_1\bar{\mathbf{u}}_1 + a_2\bar{\mathbf{u}}_2 + \ldots + a_m\bar{\mathbf{u}}_m$$

Linear combination

- Alternatively $\bar{\mathbf{v}}$ is a **linear combination** of $\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \ldots, \bar{\mathbf{u}}_m$, if there exists a solution to the vector equation,

$$\bar{\mathbf{v}} = x_1\bar{\mathbf{u}}_1 + x_2\bar{\mathbf{u}}_2 + \ldots + x_m\bar{\mathbf{u}}_m$$

- where $x_1, x_2, \ldots, x_m$ are scalars

Linear combination – Ex.

- Express

$$\bar{\mathbf{v}} = \begin{bmatrix} 1 \\ 0 \\ -6 \end{bmatrix}$$

- As a linear combination of vectors

$$\bar{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \bar{\mathbf{u}}_2 = \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}, \quad \bar{\mathbf{u}}_3 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

Linear combination – Ex.

- We seek scalars x, y, z such that

$$\begin{bmatrix} 1 \\ 0 \\ -6 \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix} + z \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

Linear combination – Ex.

- This leads to the system of equations

$$x + 2y + z = 1$$
$$2x + 5y + 3z = 0$$
$$x + 4y + 4z = -6$$

Linear combination – Ex.

- Reducing the system to **echelon form** yields

$$
\begin{aligned}
x + 2y + z &= 1 \\
y + z &= -2 \\
z &= -3 \\
\Rightarrow x = 2,\ y &= 1,\ z = -3 \\
\Rightarrow \bar{\mathbf{v}} &= 2\bar{\mathbf{u}}_1 + \bar{\mathbf{u}}_2 - 3\bar{\mathbf{u}}_3
\end{aligned}
$$

Linear combination – Ex.

- Expressing a given vector $\bar{\mathbf{v}}$ in $\mathbb{K}^n$ as a **linear combination** of vectors $\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \ldots, \bar{\mathbf{u}}_m$ in $\mathbb{K}^n$ is equivalent to solving the system

$$\begin{bmatrix} | & | & & | \\ \bar{\mathbf{u}}_1 & \bar{\mathbf{u}}_2 & \cdots & \bar{\mathbf{u}}_m \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\mathbf{U}\bar{\mathbf{x}} = \bar{\mathbf{v}}$$

Linear combination – Ex.

- As seen previously, such a system may have

  – a **unique solution, many solutions, or no solution**.

- **No solution**—means that $\bar{\mathbf{v}}$ **cannot be written** as a linear combination of the $\bar{\mathbf{u}}_i$.

Linear combination – Ex.

- Express the polynomial

$$v(t) = 4t^2 + 7t + 4$$

- as a linear combination of

$$p_1(t) = t^2 + t + 1$$
$$p_2(t) = 2t^2 + 3t + 1$$
$$p_3(t) = t^2 + 2t + 3$$

Linear combination – Ex.

$$v(t) = 4t^2 + 7t + 4$$

$$p_1(t) = t^2 + t + 1$$

$$p_2(t) = 2t^2 + 3t + 1$$

$$p_3(t) = t^2 + 2t + 3$$

- We seek scalars x, y, z such that

$$4t^2 + 7t + 4 = x(t^2 + t + 1)$$

$$+y(2t^2 + 3t + 1) + z(t^2 + 2t + 3)$$

Linear combination – Ex.

$$4t^2 + 7t + 4 = x(t^2 + t + 1)$$
$$+y(2t^2 + 3t + 1) + z(t^2 + 2t + 3)$$

- The above can be rewritten as

$$4t^2 + 7t + 4 = t^2(x + 2y + z)$$
$$+t(x + 3y + 2z) + (x + y + 3z)$$

Linear combination – Ex.

- Reduce the system to the echelon form

$$x + 2y + z = 4$$
$$y + z = 3$$
$$3z = 3$$

- The system has the solution

$$x = -1,\ y = 2,\ z = 1$$
$$\Rightarrow v(t) = -p_1(t) + 2p_2(t) + p_3(t)$$

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Spanning sets

Sets of vectors whose combinations give the whole space



Spanning set

- Let $\mathcal{V}$ be a vector space over $\mathbb{K}$

- Vectors $\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \ldots, \bar{\mathbf{u}}_m$ in $\mathcal{V}$ are said to span $\mathcal{V}$ or to form a **spanning set** of $\mathcal{V}$ if every vector $\bar{\mathbf{v}} \in V$ is a linear combination of $\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \ldots, \bar{\mathbf{u}}_m$

Spanning set

- That is there exists scalars $a_1, a_2, \ldots, a_m$ such that

$$\bar{\mathbf{v}} = a_1\bar{\mathbf{u}}_1 + a_2\bar{\mathbf{u}}_2 + \ldots + a_m\bar{\mathbf{u}}_m$$

Spanning set – Example 1

- The vectors below form a **spanning set** for $\mathbb{R}^3$

$$\bar{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{\mathbf{e}}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \bar{\mathbf{e}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Spanning set – Example 1

- Specifically, if $\bar{\mathbf{v}} = [a\ b\ c]^T$ is any vector in $\mathbb{R}^3$

$$\bar{\mathbf{v}} = a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix} + c\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$= a\bar{\mathbf{e}}_1 + b\bar{\mathbf{e}}_2 + c\bar{\mathbf{e}}_3$$

Spanning set – Example 2

- The vectors below also form a **spanning set** for $\mathbb{R}^3$

$$\bar{\mathbf{w}}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \bar{\mathbf{w}}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \bar{\mathbf{w}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Spanning set – Example 2

- Specifically, if $\bar{v} = [a\ b\ c]^T$ is any vector in $\mathbb{R}^3$

$$\bar{\mathbf{v}} = a\begin{bmatrix}1\\1\\1\end{bmatrix} + (b-a)\begin{bmatrix}0\\1\\1\end{bmatrix} + (c-b)\begin{bmatrix}0\\0\\1\end{bmatrix}$$
$$= a\bar{\mathbf{w}}_1 + (b-a)\bar{\mathbf{w}}_2 + (c-b)\bar{\mathbf{w}}_3$$

  - Thus, the **spanning set** is NOT unique

Spanning set – polynomials

- Consider the vector space $\mathcal{V} = \mathcal{P}_n(t)$ consisting of all **polynomials of degree** $\leq n$
- Any polynomial in $\mathcal{P}_n(t)$ can be expressed as a **linear combination** of

$$1,\ t,\ t^2, \ldots, t^n$$

  - Hence, it forms a spanning set

Spanning set – polynomials

- Any polynomial in $\mathcal{P}_n(t)$ can also be expressed as a **linear combination** of

$$1,\ t-c,\ (t-c)^2, \ldots, (t-c)^n$$

- Hence, the above is another **spanning set**

Spanning set – Matrices

- Consider the space $\mathbb{R}^{2\times 2}$ consisting of all 2×2 matrices.
- They can be spanned by

$$\mathbf{E}_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{E}_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{E}_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \mathbf{E}_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Summary

- A vector space is closed under addition and scalar multiplication and obeys the axioms
- Examples: $\mathbb{K}^n$, polynomials $P_n(t)$, matrices $\mathbb{K}^{m\times n}$, functions $\mathcal{F}(X)$
- $\bar{\mathbf{v}}$ is a linear combination if $\bar{\mathbf{v}} = a_1\bar{\mathbf{u}}_1 + \cdots + a_m\bar{\mathbf{u}}_m$
- Finding the coefficients means solving a linear system
- A spanning set expresses every vector of V; it is not unique



Thank You!

Questions?