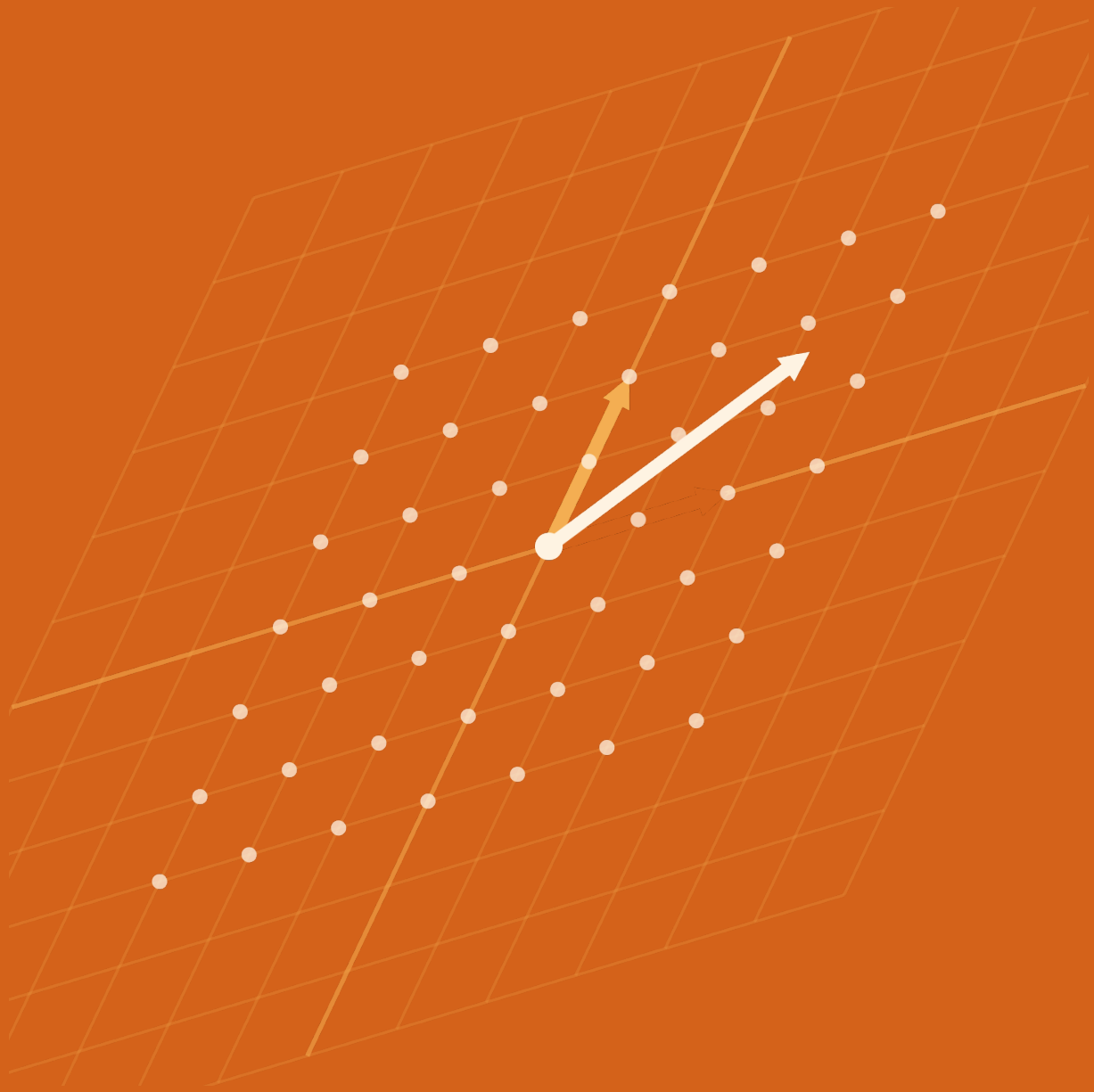
LINEAR ALGEBRA FOR AIML AND 6G

Determinants: Inverse and Other Problems



Lecture 9

Prof. Aditya K. Jagannatham
IIT Kanpur

Outline

1. **Matrix inverse**
   Classical adjoint and the inverse via cofactors

2. **Cramer’s rule**
   Solving square systems with determinants

3. **Principal minors**
   Minors whose diagonal comes from the diagonal of A

4. **Block matrices**
   Determinant of triangular block matrices

Outline

5 **Determinant and volume**
Area and volume as determinants

6 **Complex matrix**
Determinants of complex and unitary matrices

7 **Some Determinant problems**
Three worked determinant problems

01

Matrix inverse

Classical adjoint and the inverse via cofactors



Classical adjoint

- Let C_{ij} denote the **cofactor** of a_{ij}.

- The **classical adjoint** or **adjugate** of $\mathbf{A}$ is the **transpose of the matrix of cofactors**

$$\operatorname{adj}(\mathbf{A}) = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

Classical adjoint property

- Let $\mathbf{A}$ be any **square matrix**. Then

$$\mathbf{A}\ \mathrm{adj}\ (\mathbf{A}) = |\mathbf{A}|\mathbf{I}$$

- This can be demonstrated as follows

Classical adjoint property

- Consider

$$\mathbf{B} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

- Note that

$$b_{ii} = \sum_{k=1}^{n} a_{ik} C_{ik} = |\mathbf{A}|$$

Classical adjoint property

- Furthermore,

$$b_{ij} = \sum_{k=1}^{n} a_{ik} C_{jk} = 0$$

- Note that this is the determinant of the matrix obtained by **replacing jth row by the ith row**

  - Since **two rows become equal**, the determinant reduces to zero.

Classical adjoint property

- Therefore, it follows that

$$\mathbf{B} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} |\mathbf{A}| & 0 & \dots & 0 \\ 0 & |\mathbf{A}| & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & |\mathbf{A}| \end{bmatrix} = |\mathbf{A}|\,\mathbf{I}$$

Inverse of square matrix

- Let $\mathbf{A}$ be any **square matrix**. Since

$$\mathbf{A}\ \ \mathrm{adj}\,(\mathbf{A}) = \mathrm{adj}\,(\mathbf{A})\ \ \mathbf{A} = |\mathbf{A}|\,\mathbf{I}$$

$$\Rightarrow \mathbf{A}\frac{\mathrm{adj}\,(\mathbf{A})}{|\mathbf{A}|} = \mathbf{I}$$

$$\Rightarrow \mathbf{A}^{-1} = \frac{\mathrm{adj}\,(\mathbf{A})}{|\mathbf{A}|}$$

Inverse – Example

- Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ -2 & 1 & 4 \end{bmatrix}$$

$$C_{11} = \begin{vmatrix} 0 & 2 \\ 1 & 4 \end{vmatrix} = 0 - 2 = -2$$

$$C_{21} = -\begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} = (-1)(8 + 1) = -9$$

Inverse – Example

- **Classical adjoint** is

$$\text{adj}\ (\mathbf{A}) = \begin{bmatrix} -2 & -9 & 4 \\ -16 & 2 & -5 \\ 3 & -5 & -6 \end{bmatrix}$$

Inverse – Example

- Determinant (expanding along row 2) is

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ -2 & 1 & 4 \end{bmatrix}$$

$$|\mathbf{A}| = -3(8+1) - 2(1+4)$$

$$= -27 - 10 = -37$$

Inverse – Example

- Hence, its **inverse** is

$$\mathbf{A}^{-1} = \frac{\text{adj}\,(\mathbf{A})}{|\mathbf{A}|}$$

$$= -\frac{1}{37}\begin{bmatrix} -2 & -9 & 4 \\ -16 & 2 & -5 \\ 3 & -5 & -6 \end{bmatrix}$$

02

Cramer’s rule

Solving square systems with determinants



Cramer's rule

- Consider the system of equations

$$\mathbf{A}\bar{\mathbf{x}} = \bar{\mathbf{b}}$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Cramer's rule

- Let $\mathbf{A}_i$ be the matrix obtained from $\mathbf{A}$ by **replacing** the ith column of $\mathbf{A}$ by the column vector $\mathbf{b}$

$$\mathbf{A}_2 = \begin{bmatrix} a_{11} & b_1 & \dots & a_{1n} \\ a_{21} & b_2 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & b_n & \dots & a_{nn} \end{bmatrix}$$

Cramer's rule

- The system has a **unique solution** if and only if

$$|\mathbf{A}| \neq 0$$

- The **unique solution** to the system is

$$x_1 = \frac{|\mathbf{A}_1|}{|\mathbf{A}|}, x_2 = \frac{|\mathbf{A}_2|}{|\mathbf{A}|}, \ldots, x_n = \frac{|\mathbf{A}_n|}{|\mathbf{A}|}$$

Cramer's rule – Example

- Consider the system of equations

$$x + 2y + z = 3$$
$$2x - y + 3z = 14$$
$$3x + y - 2z = -1$$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 3 \\ 3 & 1 & -2 \end{bmatrix}$$

$$|\mathbf{A}| = 1(2-3) - 2(-4-9) + 1(2+3)$$
$$= -1 + 26 + 5 = 30$$

Cramer’s rule – Example

$$x + 2y + z = 3$$
$$2x - y + 3z = 14$$
$$3x + y - 2z = -1$$

Cramer's rule – Example

- $\mathbf{A}_1$ and $|\mathbf{A}_1|$ are given as

$$\mathbf{A}_1 = \begin{bmatrix} 3 & 2 & 1 \\ 14 & -1 & 3 \\ -1 & 1 & -2 \end{bmatrix}$$

$$|\mathbf{A}_1| = 3(2-3) - 2(-28+3) + 1(14-1)$$
$$= -3 + 50 + 13 = 60$$

Cramer’s rule – Example

$$x + 2y + z = 3$$
$$2x - y + 3z = 14$$
$$3x + y - 2z = -1$$

Cramer's rule – Example

- $\mathbf{A}_2$ and $|\mathbf{A}_2|$ are given as

$$\mathbf{A}_2 = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 14 & 3 \\ 3 & -1 & -2 \end{bmatrix}$$

$$|\mathbf{A}_2| = 1(-28+3) - 3(-4-9) + 1(-2-42)$$
$$= -25 + 39 - 44 = -30$$

Cramer’s rule – Example

$$x + 2y + z = 3$$
$$2x - y + 3z = 14$$
$$3x + y - 2z = -1$$

Cramer's rule – Example

- $\mathbf{A}_3$ and $|\mathbf{A}_3|$ are given as

$$\mathbf{A}_3 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 14 \\ 3 & 1 & -1 \end{bmatrix}$$

$$|\mathbf{A}_3| = 1(1-14) - 2(-2-42) + 3(2+3)$$
$$= -13 + 88 + 15 = 90$$

Cramer's rule – Example

- Therefore, **unique solution** of the system is

$$x = \frac{|\mathbf{A}_1|}{|\mathbf{A}|} = \frac{60}{30} = 2$$

$$y = \frac{|\mathbf{A}_2|}{|\mathbf{A}|} = \frac{-30}{30} = -1$$

$$z = \frac{|\mathbf{A}_3|}{|\mathbf{A}|} = \frac{90}{30} = 3$$

03

Principal minors

Minors whose diagonal comes from the diagonal of A



Principal minors

- A minor is **principal** if the **row and column indices** are the same,
- or equivalently, if the **diagonal elements of the minor** come from the diagonal of the matrix.

Principal minors

- We note that the sign of a principal minor is always $+1$,
- because the sum of the row and identical column subscripts **must always be even**.

Principal minors – Ex.

- Let $\mathbf{A}$ be defined as

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \\ 3 & 0 & 1 \end{bmatrix}$$

- Find the **principal minors** of A of orders 1, 2, and 3, respectively

Principal minors – Ex.

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \\ 3 & 0 & 1 \end{bmatrix}$$

- The principal minors of order 1 are the diagonal entries $2, 4, 1$
- There are **three principal minors** of order 2. These are

$$\begin{vmatrix} 4 & -2 \\ 0 & 1 \end{vmatrix} = 4, \quad \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = -7, \quad \begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} = 9$$

Principal minors – Ex.

- There is only **one minor** of order 3 which is the determinant of $\mathbf{A}$ itself.

$$\begin{vmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \\ 3 & 0 & 1 \end{vmatrix}$$

$$= 2(4-0) + 1(1+6) + 3(0-12)$$

$$= 8 + 7 - 36 = -21$$

04

Block matrices

Determinant of triangular block matrices



Block matrix

- Suppose $\mathbf{M}$ is an **upper (lower) triangular block** matrix with the diagonal blocks $\mathbf{A}_1; \mathbf{A}_2; \ldots; \mathbf{A}_n$.
- Then

$$|\mathbf{M}| = |\mathbf{A}_1|\,|\mathbf{A}_2| \ldots |\mathbf{A}_n|$$

Block matrix – Example

- Find $|\mathbf{M}|$ where

$$\mathbf{M} = \begin{bmatrix} 3 & 1 & 2 & -1 & 4 \\ 2 & 4 & 0 & 5 & 1 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & -2 & 1 & 3 \\ 0 & 0 & 4 & 0 & 1 \end{bmatrix}$$

Block matrix – Example

$$\mathbf{M} = \begin{bmatrix} 3 & 1 & 2 & -1 & 4 \\ 2 & 4 & 0 & 5 & 1 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & -2 & 1 & 3 \\ 0 & 0 & 4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_3 \\ \mathbf{0} & \mathbf{A}_2 \end{bmatrix}$$

- Note that $\mathbf{M}$ is an **upper triangular block matrix**

$$|\mathbf{A}_1| = \begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} = 12 - 2 = 10$$

Block matrix – Example

- Furthermore,

$$|\mathbf{A}_2| = \begin{vmatrix} 1 & 3 & 2 \\ -2 & 1 & 3 \\ 4 & 0 & 1 \end{vmatrix}$$

$$= 1(1-0) - 3(-2-12) + 2(0-4)$$

$$= 1 + 42 - 8 = 35$$

Block matrix – Example

- Since $\mathbf{M}$ is an **upper triangular block matrix**

$$|\mathbf{M}| = \begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} \begin{vmatrix} 1 & 3 & 2 \\ -2 & 1 & 3 \\ 4 & 0 & 1 \end{vmatrix}$$

$$= 10 \times 35 = 350$$

Remark

- Note that if M = [A B; C D], where A, B, C, D are square matrices.

- Then it is **not generally true** that

$$|\mathbf{M}| = |\mathbf{A}|\,|\mathbf{D}| - |\mathbf{B}|\,|\mathbf{C}|$$

05

Determinant and volume

Area and volume as determinants



Parallelogram

- Determinants are related to the notions of **area and volume** as follows.

- Let $\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \ldots, \bar{\mathbf{u}}_n$ in $\mathbb{R}^n$ determine a region *S*.

- When $n = 2$, S is a **parallelogram**

- When $n = 3$, S is a **parallelepiped**

Determinant and volume

- Let $V(S)$ denote the volume of S
- Let $\mathbf{A}$ be defined as the matrix

$$\mathbf{A} = \begin{bmatrix} | & | & | & | \\ \bar{\mathbf{u}}_1 & \bar{\mathbf{u}}_2 & \dots & \bar{\mathbf{u}}_n \\ | & | & | & | \end{bmatrix}$$

- Then

$$V(S) = |\det(\mathbf{A})|$$

Volume – Example

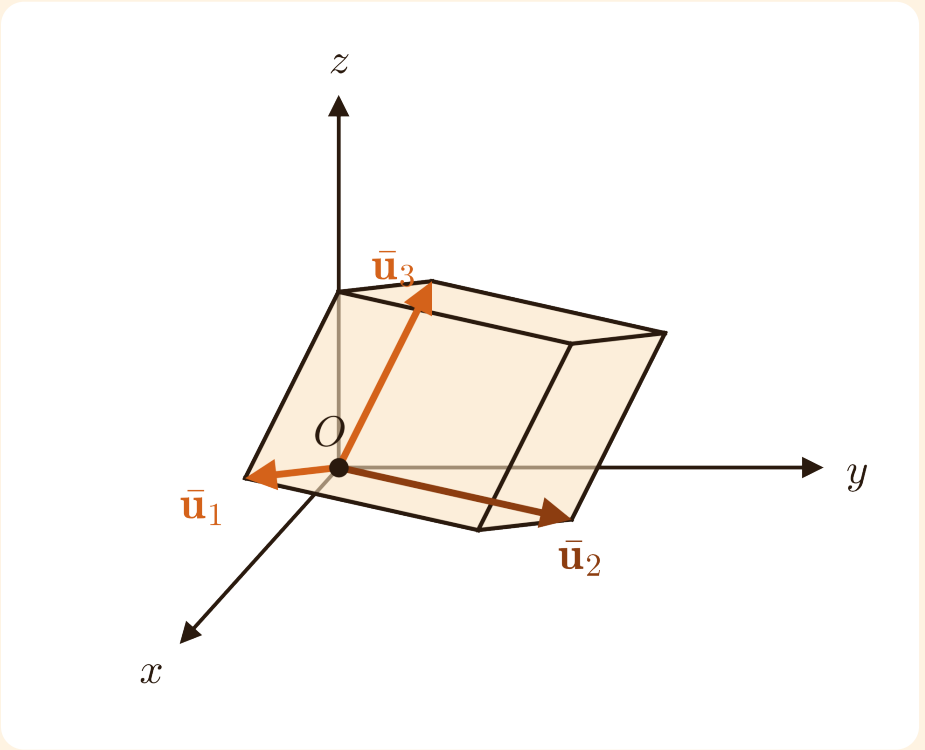
- Let $V(S)$ denote the volume of S
- Consider the vectors below

$$\bar{\mathbf{u}}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad \bar{\mathbf{u}}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \quad \bar{\mathbf{u}}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

- Find the volume $V(S)$ of the **parallelepiped** S in $\mathbb{R}^3$ determined by the three vectors.

Volume – Example

- The **parallelepiped** is shown in the figure below



Volume – Example

- The volume is given as **determinant** of matrix with columns $\bar{\mathbf{u}}_1$, $\bar{\mathbf{u}}_2$, $\bar{\mathbf{u}}_3$
- This can be computed as

$$\left| \begin{vmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 1 & 0 & 2 \end{vmatrix} \right|$$

$$= |2(6-0) - 1(0-1) + 0(0-3)|$$

$$= |12+1| = 13$$

06

Complex matrix

Determinants of complex and unitary matrices



Complex matrix

- Determinant of a complex matrix is computed similar to that of a real matrix

- It satisfies the property

$$\left|\mathbf{A}^H\right| = \left|\left(\mathbf{A}^*\right)^T\right| = \left|\mathbf{A}^*\right| = \left|\mathbf{A}\right|^*$$

Unitary matrix

- Recall, **unitary matrix** satisfies

$$\mathbf{U}\mathbf{U}^H = \mathbf{I}$$
$$\Rightarrow |\mathbf{U}|\left|\mathbf{U}^H\right| = |\mathbf{I}| = 1$$
$$\Rightarrow |\mathbf{U}|\,|\mathbf{U}|^* = 1$$
$$\Rightarrow |\det \mathbf{U}|^2 = 1$$
$$\Rightarrow |\det \mathbf{U}| = 1$$
$$\Rightarrow |\mathbf{U}| = e^{j\theta}$$

07

Some Determinant problems

Three worked determinant problems



Some Determinant problems

- We explore some problems in solving Determinants

Problem #1

- Simplify

$$\begin{vmatrix} a & a^2 & b+c \\ b & b^2 & c+a \\ c & c^2 & a+b \end{vmatrix}$$

Problem #1

- Perform $C_1 + C_3 \to C_3$

$$\begin{vmatrix} a & a^2 & b+c \\ b & b^2 & c+a \\ c & c^2 & a+b \end{vmatrix} = \begin{vmatrix} a & a^2 & a+b+c \\ b & b^2 & b+c+a \\ c & c^2 & c+a+b \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

Problem #1

- Perform now $R_1 - R_2 \to R_1$, $R_2 - R_3 \to R_2$

$$\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = \begin{vmatrix} a-b & a^2-b^2 & 0 \\ b-c & b^2-c^2 & 0 \\ c & c^2 & 1 \end{vmatrix}$$

$$= (a-b)\left(b^2-c^2\right) - (b-c)\left(a^2-b^2\right)$$

$$= (a-b)(b-c)(b+c-a-b)$$

$$= (a-b)(b-c)(c-a)$$

Problem #1

- Therefore, it follows that

$$\begin{vmatrix} a & a^2 & b+c \\ b & b^2 & c+a \\ c & c^2 & a+b \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

$$= (a+b+c)(a-b)(b-c)(c-a)$$

Problem #2

- Simplify

$$\begin{vmatrix} 1+a & 1+ax & 1+ax^2 \\ 1+b & 1+bx & 1+bx^2 \\ 1+c & 1+cx & 1+cx^2 \end{vmatrix}$$

Problem #2

- Using sum property

$$\begin{vmatrix} 1+a & 1+ax & 1+ax^2 \\ 1+b & 1+bx & 1+bx^2 \\ 1+c & 1+cx & 1+cx^2 \end{vmatrix}$$

$$= \underbrace{\begin{vmatrix} 1 & 1+ax & 1+ax^2 \\ 1 & 1+bx & 1+bx^2 \\ 1 & 1+cx & 1+cx^2 \end{vmatrix}}_{D_1} + \underbrace{\begin{vmatrix} a & 1+ax & 1+ax^2 \\ b & 1+bx & 1+bx^2 \\ c & 1+cx & 1+cx^2 \end{vmatrix}}_{D_2}$$

Problem #2

- D_1 can be simplified as

$$\begin{vmatrix} 1 & 1+ax & 1+ax^2 \\ 1 & 1+bx & 1+bx^2 \\ 1 & 1+cx & 1+cx^2 \end{vmatrix}$$

Problem #2

$$= \underbrace{\begin{vmatrix} 1 & 1 & 1+ax^2 \\ 1 & 1 & 1+bx^2 \\ 1 & 1 & 1+cx^2 \end{vmatrix}}_{0} + \begin{vmatrix} 1 & ax & 1+ax^2 \\ 1 & bx & 1+bx^2 \\ 1 & cx & 1+cx^2 \end{vmatrix}$$

$$= \underbrace{\begin{vmatrix} 1 & 1 & 1+ax^2 \\ 1 & 1 & 1+bx^2 \\ 1 & 1 & 1+cx^2 \end{vmatrix}}_{0} + \begin{vmatrix} 1 & ax & ax^2 \\ 1 & bx & bx^2 \\ 1 & cx & cx^2 \end{vmatrix}$$

Problem #2

- Further, on performing $C_3 - xC_2 \rightarrow C_3$, the last determinant reduces to

$$\begin{vmatrix} 1 & ax & ax^2 \\ 1 & bx & bx^2 \\ 1 & cx & cx^2 \end{vmatrix} = \begin{vmatrix} 1 & ax & 0 \\ 1 & bx & 0 \\ 1 & cx & 0 \end{vmatrix} = 0$$

Problem #2

- Coming next to D_2

$$\begin{vmatrix} a & 1+ax & 1+ax^2 \\ b & 1+bx & 1+bx^2 \\ c & 1+cx & 1+cx^2 \end{vmatrix}$$

$$= \begin{vmatrix} a & 1 & 1+ax^2 \\ b & 1 & 1+bx^2 \\ c & 1 & 1+cx^2 \end{vmatrix} + \underbrace{\begin{vmatrix} a & ax & 1+ax^2 \\ b & bx & 1+bx^2 \\ c & cx & 1+cx^2 \end{vmatrix}}_{C_2 - xC_1 = 0 \to C_2}$$

Problem #2

- Simplify now the first determinant

$$\begin{vmatrix} a & 1 & 1+ax^2 \\ b & 1 & 1+bx^2 \\ c & 1 & 1+cx^2 \end{vmatrix}$$

$$= \underbrace{\begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix}}_{0} + \underbrace{\begin{vmatrix} a & 1 & ax^2 \\ b & 1 & bx^2 \\ c & 1 & cx^2 \end{vmatrix}}_{C_3 - x^2 C_1 = 0 \to C_3} = 0$$

Problem #2

- Therefore, it follows that

$$\begin{vmatrix} 1+a & 1+ax & 1+ax^2 \\ 1+b & 1+bx & 1+bx^2 \\ 1+c & 1+cx & 1+cx^2 \end{vmatrix} = 0$$

Problem #3

- Consider the matrix $\mathbf{A}$ below

$$\mathbf{A} = \begin{bmatrix} x+1 & x+2 & x+3 \\ x & x+3 & x+3 \\ x & x+2 & x+4 \end{bmatrix}$$

$$If\ |\mathbf{A}| = 99,\ x = ?$$

Problem #3

- Perform $R_1 - R_3 \rightarrow R_1$, $R_2 - R_3 \rightarrow R_2$

$$|\mathbf{A}| = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ x & x+2 & x+4 \end{vmatrix}$$

$$x + 4 + x + 2 + x = 99$$

$$\Rightarrow 3x + 6 = 99$$

$$\Rightarrow x = \frac{93}{3} = 31$$

Summary

- $\mathrm{adj}(\mathbf{A})$ is the transpose of the cofactor matrix
- $\mathbf{A}\,\mathrm{adj}(\mathbf{A}) = |\mathbf{A}|\mathbf{I}$, so $\mathbf{A}^{-1} = \mathrm{adj}(\mathbf{A})/|\mathbf{A}|$
- Cramer's rule: $x_i = |\mathbf{A}_i|/|\mathbf{A}|$ when $|\mathbf{A}| \neq 0$
- Triangular block matrix: $|\mathbf{M}| = |\mathbf{A}_1||\mathbf{A}_2|\cdots|\mathbf{A}_n|$
- Volume of the parallelepiped spanned by the columns is $|\det(\mathbf{A})|$
- $|\mathbf{A}^H| = |\mathbf{A}|^*$; a unitary matrix has $|\det \mathbf{U}| = 1$



Thank You!

Questions?