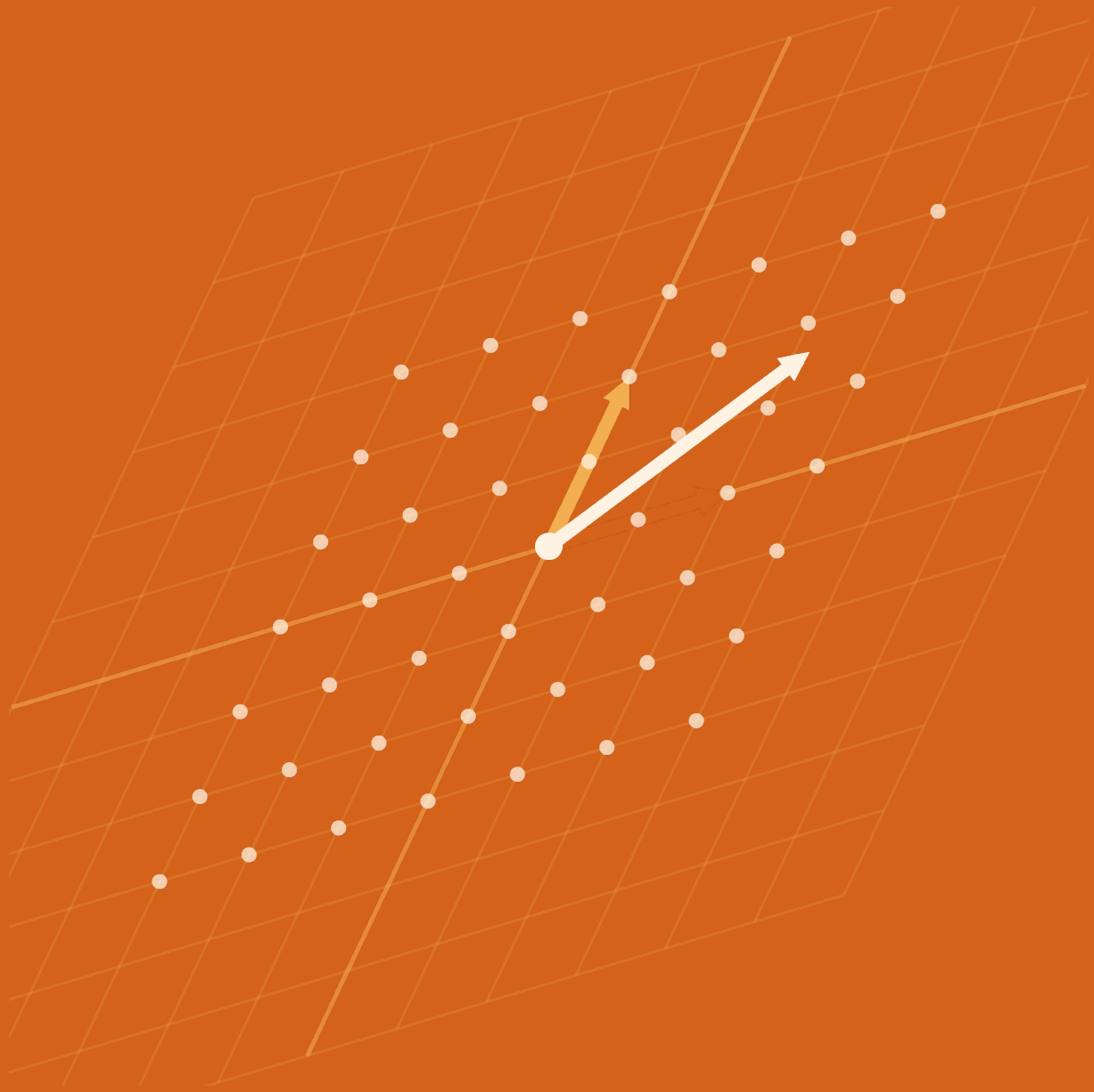
LINEAR ALGEBRA FOR AIML AND 6G

Determinants: Definition and Properties



Lecture 8

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Outline

1. **Determinant**
   Definition, small orders and solving two equations

2. **Minors and cofactors**
   Minors, cofactors and the Laplace expansion

3. **Determinant properties**
   How row operations, transposes and products affect the determinant

4. **Invertibility and determinant**
   A matrix is invertible exactly when its determinant is nonzero

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Determinant

Definition, small orders and solving two equations



Determinant

- Each $n \times n$ square matrix $\mathbf{A} = [a_{ij}]$ is assigned a **special scalar** called the **determinant** of $\mathbf{A}$
  - denoted by $\det(\mathbf{A})$ **or** $|\mathbf{A}|$

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

Determinant of order 1

- Determinant of order 1 is the **scalar itself**

$$\det\,(a_{11}) = a_{11}$$

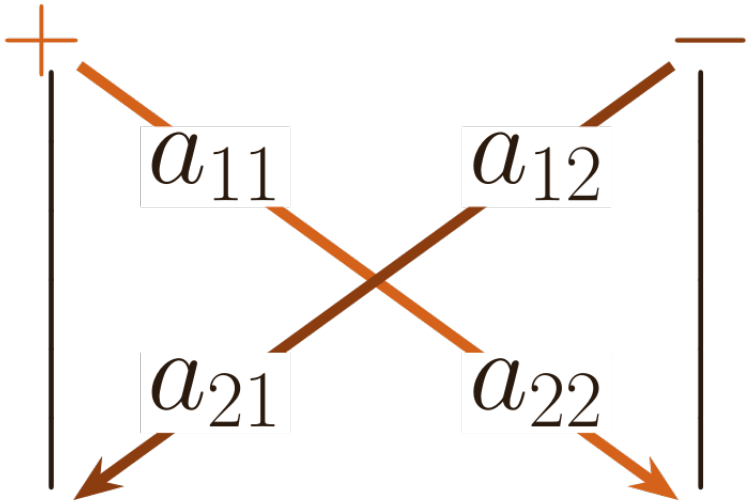
Determinant of order 2

- Determinant of order 2 is

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Determinant of order 2

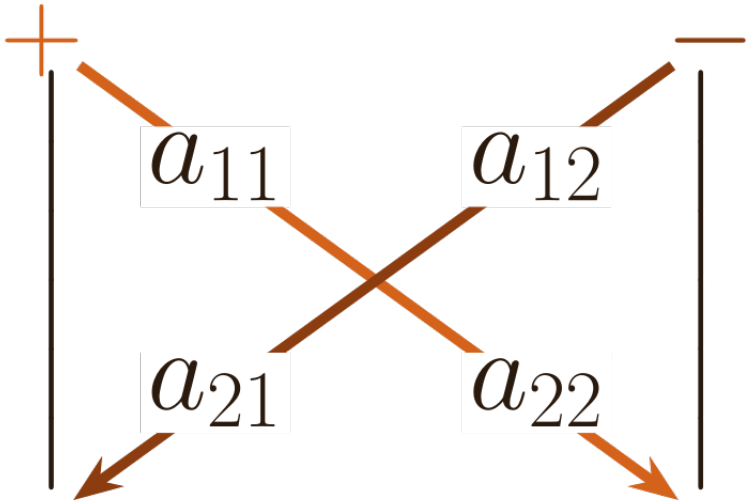
- The **determinant** of order two may easily be remembered by using the following diagram:



$$a_{11}a_{22} - a_{12}a_{21}$$

Determinant of order 2

- The determinant is equal to the **product of the elements** along the plus-labeled arrow
  - **minus** the product of the elements along the minus-labeled arrow



$$a_{11}a_{22} - a_{12}a_{21}$$

Determinant – Examples

- Determinant of order 1

$$\det(15) = 15$$
$$\det(-9) = -9$$
$$\det(t+5) = t+5$$

Determinant – Examples

- Determinant of order 2

$$\begin{vmatrix} 7 & 2 \\ 3 & 4 \end{vmatrix} = 7 \times 4 - 2 \times 3$$

$$= 28 - 6 = 22$$

$$\begin{vmatrix} 6 & -1 \\ 3 & 2 \end{vmatrix} = 12 + 3 = 15$$

Application to linear equations

- Consider two **linear equations** in two unknowns, say

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$
$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

- The system has a **unique solution** if and only if $D \neq 0$.

Application to linear equations

- Solution can be expressed in terms of **determinants** as

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Application to linear Eqns. – Ex.

- Consider the system

$$3x + 2y = 8$$
$$5x - 4y = 6$$

- The solution is given as

$$x = \frac{\begin{vmatrix} 8 & 2 \\ 6 & -4 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 5 & -4 \end{vmatrix}} = \frac{-44}{-22} = 2$$

Application to linear Eqns. – Ex.

- Consider the system

$$3x + 2y = 8$$
$$5x - 4y = 6$$

- The solution is given as

$$y = \frac{\begin{vmatrix} 3 & 8 \\ 5 & 6 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 5 & -4 \end{vmatrix}} = \frac{-22}{-22} = 1$$

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Minors and cofactors

Minors, cofactors and the Laplace expansion



Minor

- Consider an $n \times n$ matrix.
- Let M_{ij} denote the $(n-1) \times (n-1)$ **submatrix** of $\mathbf{A}$
  - obtained by deleting its ith row and jth column.
- The determinant $|M_{ij}|$ is called the **minor** of the element a_{ij} of $\mathbf{A}$

Cofactor

- The **cofactor** of a_{ij}, denoted by C_{ij}, is the **signed minor**

$$C_{ij} = (-1)^{i+j} \, |M_{ij}|$$

Example

- Consider the matrix below

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 4 \\ 3 & 5 & 1 \\ -2 & 6 & 7 \end{bmatrix}$$

- $|M_{23}|$ and C_{23} are below

$$|\mathbf{M}_{23}| = \begin{vmatrix} 2 & -1 \\ -2 & 6 \end{vmatrix} = 12 - 2 = 10$$

$$C_{23} = (-1)^{2+3} \times 10 = -10$$

Determinant formula

- The **Laplace formula** for the determinant of a square matrix $\mathbf{A} = [a_{ij}]$ is equal to

  – the **sum of the products** obtained by multiplying the elements of any row (column) by their **respective cofactors**

Determinant formula

- This is mathematically expressed as below

$$|\mathbf{A}| = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij}$$

$$|\mathbf{A}| = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj} = \sum_{i=1}^{n} a_{ij}C_{ij}$$

Determinant – Example

- Evaluate the determinant below

$$\begin{vmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \\ 1 & 3 & 2 \end{vmatrix}$$

$$= 2\begin{vmatrix} -2 & 4 \\ 3 & 2 \end{vmatrix} - 1\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix}$$

$$= -32 - 2 - 11 = -45$$

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Determinant properties



How row operations, transposes and products affect the determinant

Determinant of Transpose

- The determinant of matrix $\mathbf{A}$ and its **transpose** $\mathbf{A}^T$ are equal

$$|\mathbf{A}| = \left|\mathbf{A}^T\right|$$

Zero row

- Let $\mathbf{A}$ be a square matrix
- If $\mathbf{A}$ has a **row (column) of zeros**, then

$$|\mathbf{A}| = 0$$

Identical rows

- If $\mathbf{A}$ has **two identical rows** (columns), then

$$|\mathbf{A}| = 0$$

Triangular/ Diagonal

- If $\mathbf{A}$ is a **lower or upper triangular matrix or diagonal matrix**,
  - then $|\mathbf{A}|$ equals the product of diagonal elements

$$|\mathbf{A}| = a_{11} \times a_{22} \times \ldots \times a_{nn}$$

Identity matrix

- In particular $|\mathbf{I}|$ is given as

$$|\mathbf{I}| = 1 \times 1 \times \ldots \times 1 = 1$$

Interchange

- If two rows (columns) of $\mathbf{A}$ are **interchanged** to obtain $\mathbf{B}$

$$|\mathbf{B}| = -|\mathbf{A}|$$

Interchange – Example

$$\begin{vmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \\ 1 & 3 & 2 \end{vmatrix} = -45$$

Interchange – Example

- Now **interchange** first two columns

$$\begin{vmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \\ 1 & 3 & 2 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & 2 & -1 \\ -2 & 3 & 4 \\ 3 & 1 & 2 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} - 2 \begin{vmatrix} -2 & 4 \\ 3 & 2 \end{vmatrix} + (-1) \begin{vmatrix} -2 & 3 \\ 3 & 1 \end{vmatrix}$$

$$= 2 + 32 + 11 = 45$$

Scalar multiple

- If a row (column) of $\mathbf{A}$ is **multiplied by a scalar** k to obtain $\mathbf{B}$,

$$|\mathbf{B}| = k\,|\mathbf{A}|$$

Scalar multiple – example

$$\begin{vmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \\ 1 & 3 & 2 \end{vmatrix} = -45$$

Scalar multiple – example

- Let us multiply first row by 2

$$\begin{vmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \\ 1 & 3 & 2 \end{vmatrix} \rightarrow \begin{vmatrix} 4 & 2 & -2 \\ 3 & -2 & 4 \\ 1 & 3 & 2 \end{vmatrix}$$

$$= 4 \begin{vmatrix} -2 & 4 \\ 3 & 2 \end{vmatrix} - 2 \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} + (-2) \begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix}$$

$$= -64 - 4 - 22 = -90 = -45 \times 2$$

Scalar multiple – Proof

- Scale jth column by k
- Determinant becomes

$$ka_{1j}C_{1j} + ka_{2j}C_{2j} + \ldots + ka_{nj}C_{nj} = \sum_{i=1}^{n} ka_{ij}C_{ij} = k\,|\mathbf{A}|$$

Row operation

- If a **multiple of a row** (column) of $\mathbf{A}$ were added to another row (column) of $\mathbf{A}$ to obtain $\mathbf{B}$, $R_i + kR_j \rightarrow R_i$

$$|\mathbf{B}| = |\mathbf{A}|$$

Row operation

$$\begin{vmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \\ 1 & 3 & 2 \end{vmatrix} = -45$$

Row operation

- Perform $R_1 + 2R_2 \to R_1$

$$\begin{vmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \\ 1 & 3 & 2 \end{vmatrix} \to \begin{vmatrix} 8 & -3 & 7 \\ 3 & -2 & 4 \\ 1 & 3 & 2 \end{vmatrix}$$

$$= 8 \begin{vmatrix} -2 & 4 \\ 3 & 2 \end{vmatrix} + 3 \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} + 7 \begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix}$$

$$= -128 + 6 + 77 = -45$$

Sum property

- Determinant behaves like a **linear function** on the rows (columns) of the matrix

$$\begin{vmatrix} a_{11}+\tilde{a}_{11} & a_{12} & a_{13} \\ a_{21}+\tilde{a}_{21} & a_{22} & a_{23} \\ a_{31}+\tilde{a}_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} \tilde{a}_{11} & a_{12} & a_{13} \\ \tilde{a}_{21} & a_{22} & a_{23} \\ \tilde{a}_{31} & a_{32} & a_{33} \end{vmatrix}$$

Sum property – Proof

- The sum property can be proved as below

$$|\mathbf{A}| = \sum_{j=1}^{n} (a_{ij} + \tilde{a}_{ij}) C_{ij}$$

$$= \sum_{j=1}^{n} a_{ij} C_{ij} + \sum_{j=1}^{n} \tilde{a}_{ij} C_{ij}$$

Product property

- Let $\mathbf{A}$, $\mathbf{B}$ be two square matrices
- **Determinant of the product**

$$|\mathbf{AB}| = |\mathbf{A}|\,|\mathbf{B}|$$

Orthogonal matrix

- Let $\mathbf{U}$ be an orthogonal matrix
- It follows that

$$\mathbf{U}\mathbf{U}^T = \mathbf{I}$$
$$\Rightarrow \left|\mathbf{U}\mathbf{U}^T\right| = |\mathbf{I}| = 1$$
$$\Rightarrow |\mathbf{U}|\left|\mathbf{U}^T\right| = 1$$
$$\Rightarrow |\mathbf{U}|^2 = 1$$
$$\Rightarrow |\mathbf{U}| = \pm 1$$

Row multiple

- Consider matrix $\mathbf{A}$ such that row (column) is **scalar multiple of another row** (column)

$$R_i = \alpha R_j$$

- Then

$$|\mathbf{A}| = 0$$

Row multiple – Proof

- This follows because one can perform row operation

$$R_i - \alpha R_j = 0 \rightarrow R_i$$

- **Determinant is zero**, since a row (column) is zero

Linear combination

- Consider matrix $\mathbf{A}$ such that row (column) is **linear combination of other rows** (column)

$$R_i = \alpha_1 R_{j_1} + \alpha_2 R_{j_2} + \ldots + \alpha_p R_{j_p}$$

- Then

$$|\mathbf{A}| = 0$$

Linear combination

- Because one can perform row (column) operation

$$R_i - \left(\alpha_1 R_{j_1} + \alpha_2 R_{j_2} + \ldots + \alpha_p R_{j_p}\right) = 0 \rightarrow R_i$$

- **Determinant is zero**, since a row (column) is zero

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Invertibility and determinant

A matrix is invertible exactly when its determinant is nonzero



Invertibility

- Matrix $\mathbf{A}$ is **invertible** if and only if

$$|\mathbf{A}| \neq 0$$

Invertibility – Proof

- We have seen that an invertible matrix $\mathbf{A}$ can be reduced to $\mathbf{I}$ using **row operations**

- Consider the **row operations**

$$\alpha R_i \rightarrow R_i$$

- In this case, the determinant **multiplied** by α

Invertibility – Proof

$$R_i + \alpha R_j \rightarrow R_i$$

- In this case determinant remains **unchanged** (multiplied by 1)

$$R_i \leftrightarrow R_j$$

- Determinant multiplied by -1

Invertibility – Proof

- Therefore, **row operations** only result in multiplication of determinant by a **non-zero constant**

- Either multiplied by α or 1 or -1

Invertibility – Proof

- An invertible matrix $\mathbf{A}$ can be reduced to $\mathbf{I}$ using **row operations**
- Let β denote the product of all **non-zero constants** multiplying $|\mathbf{A}|$ as a result of the row operations
- Therefore,

$$\beta\,|\mathbf{A}| = |\mathbf{I}| = 1$$

$$\Rightarrow |\mathbf{A}| = \frac{1}{\beta}$$

Non-invertible

- On the other hand if $\mathbf{A}$ is **non-invertible**
- **Row canonical form** of $\mathbf{A}$ is $\mathbf{A}_{eq}$, which contains at least one row of zeros

Non-invertible

- Therefore,

$$|\mathbf{A}_{eq}| = 0$$
$$\Rightarrow \beta\,|\mathbf{A}| = 0 \Rightarrow |\mathbf{A}| = 0$$

- since $\beta \neq 0$ as seen earlier

Summary

- Order 2: $|\mathbf{A}| = a_{11}a_{22} - a_{12}a_{21}$
- Cofactor $C_{ij} = (-1)^{i+j}|\mathbf{M}_{ij}|$; expand along any row or column
- Swapping rows negates $|\mathbf{A}|$; scaling a row by k scales it by k
- Adding a multiple of one row to another leaves $|\mathbf{A}|$ unchanged
- $|\mathbf{A}^T| = |\mathbf{A}|$, $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$; triangular: product of diagonal
- $\mathbf{A}$ is invertible if and only if $|\mathbf{A}| \neq 0$



Thank You!

Questions?