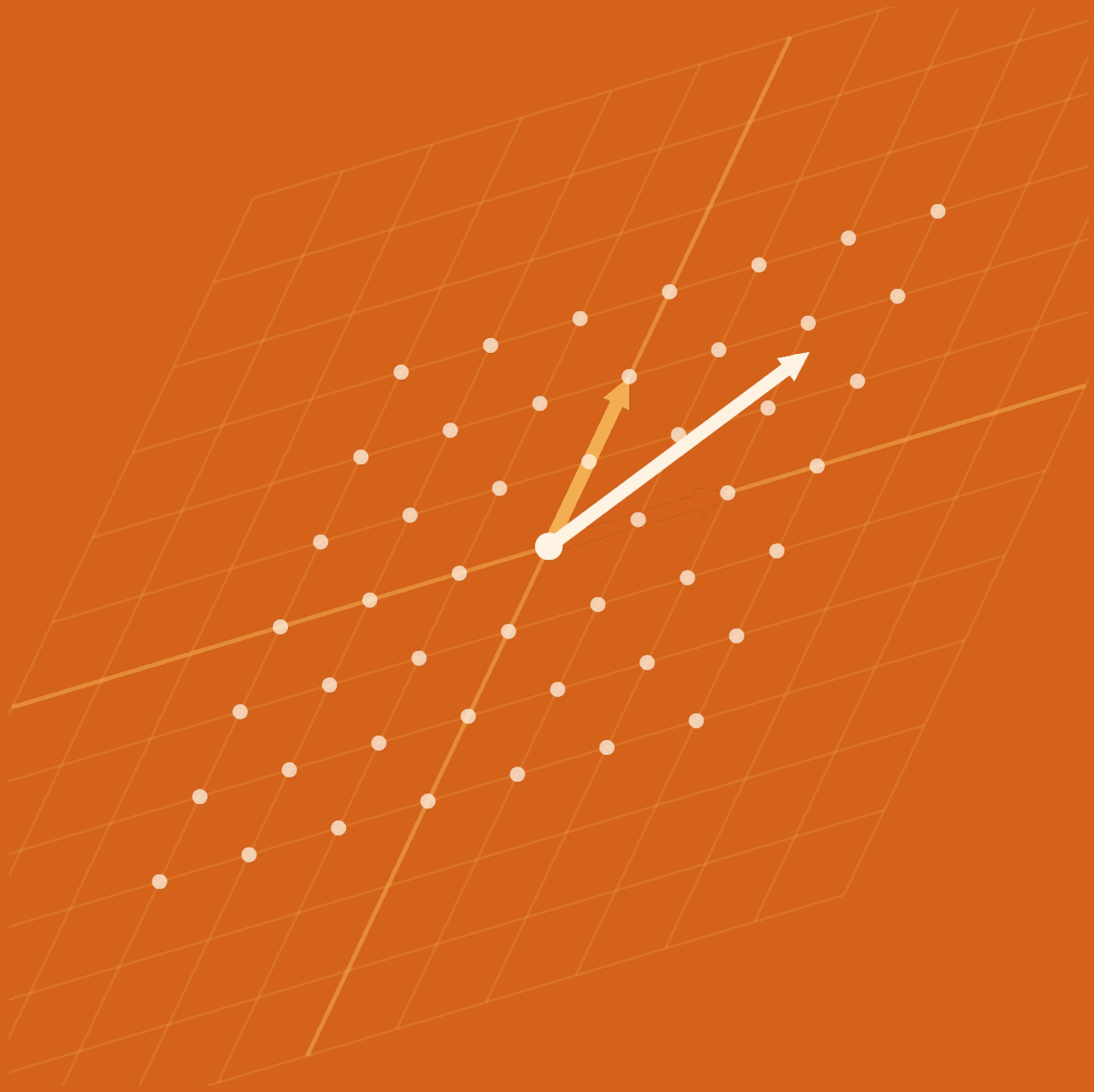
LINEAR ALGEBRA FOR AIML AND 6G

# System of Linear Equations: Echelon and Canonical forms



Lecture 7

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# Outline

1. **Echelon and row canonical forms**
   Row operations, echelon form, row canonical form and rank

2. **Solution via row operations**
   Solving systems with the augmented matrix and rank

3. **Square system of equations**
   Square systems, the inverse and Gauss-Jordan inversion

# 01

# Echelon and row canonical forms

Row operations, echelon form, row canonical form and rank



# Elementary row operations

- Consider a matrix $\mathbf{A}$ with rows $R_1$, $R_2$,…, $R_m$
  - The following operations are called **elementary row operations**

# Elementary row operations

- **Interchange** two rows $R_i$ and $R_j$.

$$R_i \leftrightarrow R_j$$

- Replace a row $R_i$ by a nonzero multiple of itself

$$kR_i \rightarrow R_i$$

# Elementary row operations

- Replace row $R_j$ by the **sum of a multiple of another row $R_i$ and itself**.

$$kR_i + R_j \to R_j$$

- One can also combine both previous steps to get

$$kR_i + k'R_j \to R_j$$

# Row operations illustration

- Consider the matrix below

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & -2 & 2 & 1 \\ 2 & 6 & -2 & 10 & 6 \\ 3 & 9 & -3 & 15 & 6 \end{bmatrix}$$

# Row operations illustration

$$-2R_1 + R_2 \to R_2$$
$$-3R_1 + R_3 \to R_3$$

- We perform the above operations which leads to the matrix

$$\mathbf{A} \sim \begin{bmatrix} 1 & 3 & -2 & 2 & 1 \\ 0 & 0 & 2 & 6 & 4 \\ 0 & 0 & 3 & 9 & 3 \end{bmatrix}$$

# Row operations illustration

$$-\frac{3}{2}R_2 + R_3 \to R_3$$

- We now perform the above operation to obtain

$$\mathbf{A} \sim \begin{bmatrix} 1 & 3 & -2 & 2 & 1 \\ 0 & 0 & 2 & 6 & 4 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix}$$

  - The above matrix is in **Echelon form**

# Echelon form

- A matrix A is called an **echelon matrix**, or is said to be in **echelon form**
  - All **zero rows**, if any, are at the bottom of the matrix.
  - Each leading nonzero entry in a row is to the **right of the leading nonzero** entry in the preceding row

# Echelon form - Example

- Matrix below is an **echelon matrix**

$$\begin{bmatrix} 0 & 3 & -1 & 2 & 6 & 4 & 0 & 5 \\ 0 & 0 & 0 & 4 & 1 & -2 & 3 & 7 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# Echelon form - Example

- The leading nonzero entry in each row in the **Echelon matrix** is termed the **PIVOT**.
  - The **pivots** in the matrix below are highlighted

$$\begin{bmatrix} 0 & \mathbf{3} & -1 & 2 & 6 & 4 & 0 & 5 \\ 0 & 0 & 0 & \mathbf{4} & 1 & -2 & 3 & 7 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{6} & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{9} & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# Row operations continued

$$\mathbf{A} \sim \begin{bmatrix} 1 & 3 & -2 & 2 & 1 \\ 0 & 0 & 2 & 6 & 4 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix}$$

- The **Echelon matrix** above can be reduced further
- Perform the operations

$$-\frac{1}{3}R_3 \to R_3$$

$$-4R_3 + R_2 \to R_2$$

$$-R_3 + R_1 \to R_1$$

# Row operations continued

- This leads to the equivalent matrix

$$\mathbf{A} \sim \begin{bmatrix} 1 & 3 & -2 & 2 & 0 \\ 0 & 0 & 2 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

# Row operations continued

- Perform now the operations

$$\frac{1}{2}R_2 \to R_2$$

$$2R_2 + R_1 \to R_1$$

# Row operations continued

- This leads to the matrix

$$\mathbf{A} \sim \begin{bmatrix} 1 & 3 & 0 & 8 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

  - This is termed the **row canonical form**

# Row canonical form

- A matrix A is said to be in **row canonical form** (or row-reduced echelon form) if it is an **echelon matrix**
- and if it satisfies the following two additional properties:
  - **Each pivot** (leading nonzero entry) is **equal to 1**.
  - Each pivot is the only **nonzero entry in its column**.

# Row canonical form – Ex.

- Matrix below is in **row canonical form**

$$\begin{bmatrix} 0 & 1 & -2 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{bmatrix}$$

# Row canonical equivalence

- Every matrix $\mathbf{A}$ is row equivalent to a unique matrix in **row canonical form**

- This is denoted by

$$\mathbf{A} \sim \mathbf{B}$$

# Rank of a matrix

- The rank of a matrix $\mathbf{A}$, written as $\mathrm{rank}(\mathbf{A})$,
  - is equal to the number of pivots in an echelon form of $\mathbf{A}$

$$\mathrm{rank}\,(\mathbf{A}) = \text{ number of pivots}$$

# Rank of a matrix – Ex.

- Consider

$$\mathbf{A} \sim \begin{bmatrix} \mathbf{1} & 3 & 0 & 8 & 0 \\ 0 & 0 & \mathbf{1} & 3 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$$

  - It can be seen that number of pivots in $\mathbf{A}$ equals $3$

- Therefore, its rank is

$$\text{rank}\,(\mathbf{A}) = 3$$

# 02

# Solution via row operations

Solving systems with the augmented matrix and rank



# Solution via row operations

- One way to solve a system of linear equations is by working with its **augmented matrix** $\mathbb{M}$

- Any **elementary row operation** on the **augmented matrix** $\mathbb{M}$ of the system

  - is equivalent to applying the corresponding operation on the system itself.

# Ex. 1 – Unique solution

- Consider the following system

$$x + 2y - z = 6$$
$$2x - y + 3z = -3$$
$$3x + y + z = 4$$

- The augmented matrix can be constructed as

$$\begin{bmatrix} 1 & 2 & -1 & 6 \\ 2 & -1 & 3 & -3 \\ 3 & 1 & 1 & 4 \end{bmatrix}$$

# Ex. 1 – Unique solution

- The augmented matrix can be reduced as

$$\begin{bmatrix} 1 & 2 & -1 & 6 \\ 2 & -1 & 3 & -3 \\ 3 & 1 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

# Ex. 1 – Unique solution

- Thus, the equivalent system follows as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

- Therefore, the solution is

$$\begin{aligned} x &= 1 \\ y &= 2 \\ z &= -1 \end{aligned}$$

# Unique solution property

- Examine again the equivalent augmented matrix above

$$\begin{bmatrix} \mathbf{1} & 0 & 0 & 1 \\ 0 & \mathbf{1} & 0 & 2 \\ 0 & 0 & \mathbf{1} & -1 \end{bmatrix}$$

- It can be seen that

$$\text{rank}\,(\mathbf{A}) = \text{rank}\,(\mathbf{M}) = n = 3$$

# Unique solution property

- Consider a system of linear equations in $n$ unknowns with **augmented matrix**

$$\mathbf{M} = \left[\mathbf{A}, \bar{\mathbf{b}}\right]$$

- Then, the system has **a solution** if and only if $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{M})$
- The solution is **unique** if

$$\text{rank}\,(\mathbf{A}) = \text{rank}\,(\mathbf{M}) = n$$

# Ex. 2 – Infinitely many soln

- Consider the following system

$$
\begin{aligned}
x_1 + 2x_2 - x_3 + 3x_4 &= 4 \\
2x_1 + 4x_2 - x_3 + 2x_4 &= 5 \\
3x_1 + 6x_2 - x_3 + x_4 &= 6
\end{aligned}
$$

- The augmented matrix is given as

$$
\begin{bmatrix}
1 & 2 & -1 & 3 & 4 \\
2 & 4 & -1 & 2 & 5 \\
3 & 6 & -1 & 1 & 6
\end{bmatrix}
$$

# Ex. 2 – Infinitely many soln

- This can be reduced to the **row canonical form** as follows

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 1 \\ 0 & 0 & 1 & -4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- The zero row is omitted to obtain the equivalent system

$$x_1 + 2x_2 - x_4 = 1$$
$$x_3 - 4x_4 = -3$$

# Ex. 2 – Infinitely many soln

- The quantities $x_2$, $x_4$, do not correspond to pivots
  - These are termed **Free variables (independent variables)**
- The quantities $x_1$, $x_3$, correspond to pivots
  - These are termed **Basic (dependent) variables**

# Ex. 2 – Infinitely many soln

- The general solution can be written in terms of the free variables as

$$x_1 = 1 - 2x_2 + x_4$$
$$x_3 = -3 + 4x_4$$

- Thus, there are **infinitely many solutions**
  - one for each set of arbitrary values for the **free variables**

# Infinite solutions property

- Examine again the equivalent augmented matrix for this system

$$\begin{bmatrix} \mathbf{1} & 2 & 0 & -1 & 1 \\ 0 & 0 & \mathbf{1} & -4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

  - The pivots are highlighted

- It can be seen that

$$\text{rank}\,(\mathbf{A}) = \text{rank}\,(\mathbf{M}) = 2 < n = 4$$

# Infinite solutions property

- Consider a system of linear equations in $n$ unknowns with **augmented matrix**

$$\mathbf{M} = [\mathbf{A}, \mathbf{B}]$$

- Then, the system has **infinitely many solutions** if and only if

$$\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{M}) < n$$

# Ex. 3 – No solution

- Consider the following system

$$
\begin{aligned}
x_1 + 2x_2 + x_3 - x_4 &= 3 \\
2x_1 + 5x_2 + 5x_3 + x_4 &= 2 \\
3x_1 + 8x_2 + 9x_3 + 3x_4 &= -5
\end{aligned}
$$

- The augmented matrix is

$$
\left[\begin{array}{ccccc}
1 & 2 & 1 & -1 & 3 \\
2 & 5 & 5 & 1 & 2 \\
3 & 8 & 9 & 3 & -5
\end{array}\right]
$$

# Ex. 3 – No solution

- The augmented matrix can be reduced via row operations to

$$\begin{bmatrix} 1 & 2 & 1 & -1 & 3 \\ 0 & 1 & 3 & 3 & -4 \\ 0 & 0 & 0 & 0 & -6 \end{bmatrix}$$

- Note that the last row corresponds to the inconsistent equation

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = -6$$

  - Thus, there is **no solution**.

# No solution property

- Examine again the equivalent augmented matrix for this system

$$\begin{bmatrix} \mathbf{1} & 2 & 1 & -1 & 3 \\ 0 & \mathbf{1} & 3 & 3 & -4 \\ 0 & 0 & 0 & 0 & \mathbf{-6} \end{bmatrix}$$

  - The pivots are highlighted

- It can be seen that

$$\text{rank}\,(\mathbf{A}) = 2 < \text{rank}\,(\mathbf{M}) = 3$$

# No solution property

- Consider a system of linear equations in $n$ unknowns with **augmented matrix**

$$\mathbf{M} = \left[\mathbf{A}, \bar{\mathbf{b}}\right]$$

- Then, the system has **no solution** if and only if

$$\text{rank}\,(\mathbf{A}) < \text{rank}\,(\mathbf{M})$$

# Solution properties

- The properties of the solution for a linear system are **summarized** below

- The system has **no solution** if and only if

$$\text{rank}\,(\mathbf{A}) < \text{rank}\,(\mathbf{M})$$

- The system has **a solution** if and only if

$$\text{rank}\,(\mathbf{A}) = \text{rank}\,(\mathbf{M})$$

# Solution properties

- Further, the solution is unique if

$$\text{rank}\,(\mathbf{A}) = \text{rank}\,(\mathbf{M}) = n$$

- And there are infinitely many solutions if

$$\text{rank}\,(\mathbf{A}) = \text{rank}\,(\mathbf{M}) < n$$

# 03

# Square system of equations

Square systems, the inverse and Gauss-Jordan inversion



# Square system

- In a **square system** the number of equations equals that of unknowns

- Consider a system of $n$ linear equations and $n$ unknowns.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\mathbf{A}\bar{\mathbf{x}} = \bar{\mathbf{b}}$$

# Square system solution

- Such a system has **unique solution** only if $\mathbf{A}$ is invertible
- The solution is given as

$$\begin{aligned} \mathbf{A}\bar{\mathbf{x}} &= \bar{\mathbf{b}} \\ \Rightarrow \mathbf{A}^{-1}\mathbf{A}\bar{\mathbf{x}} &= \mathbf{A}^{-1}\bar{\mathbf{b}} \\ \Rightarrow \bar{\mathbf{x}} &= \mathbf{A}^{-1}\bar{\mathbf{b}} \end{aligned}$$

# Square system - Example

- Consider the system

$$x + y + 2z = 3$$
$$x + 2y + 4z = 4$$
$$2x + 3y + 7z = 8$$

- This can be represented as

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 4 \\ 2 & 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix}$$

# Square system - Example

- $\mathbf{A}^{-1}$ can be calculated as

$$\mathbf{A}^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & -2 \\ -1 & -1 & 1 \end{bmatrix}$$

- Therefore, the solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{A}^{-1}\bar{\mathbf{b}} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & -2 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

# Square matrix inverse

- The **inverse** of an $n \times n$ square matrix can be found as follows

- Form the $n \times 2n$ block matrix

$$\mathbf{M} = [\mathbf{A}|\mathbf{I}]$$

# Square matrix inverse

- Row reduce $\mathbf{M}$ to its **canonical form**

$$\mathbf{M} = [\mathbf{I}|\mathbf{B}]$$

- $\mathbf{A}^{-1}$ can be obtained as

$$\mathbf{A}^{-1} = \mathbf{B}$$

# Matrix inverse - Example

- Consider $\mathbf{A}$ given as

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$$

- **Augmented matrix** with identity can be constructed as below

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{bmatrix}$$

# Matrix inverse - Example

- The **Augmented matrix** can be reduced to

$$\mathbf{M} \sim \begin{bmatrix} 1 & 0 & 0 & -2 & 2 & -1 \\ 0 & 1 & 0 & -5 & 3 & -1 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{bmatrix}$$

- Therefore,

$$\mathbf{A}^{-1} = \begin{bmatrix} -2 & 2 & -1 \\ -5 & 3 & -1 \\ 3 & -2 & 1 \end{bmatrix}$$

# Summary

- Row operations: $R_i \leftrightarrow R_j$, $kR_i \to R_i$, $kR_i + R_j \to R_j$
- Echelon form: zero rows at the bottom, pivots move right
- Row canonical form: pivots equal 1 and are alone in their columns
- $\mathrm{rank}(\mathbf{A})$ = number of pivots in an echelon form
- Solution exists iff $\mathrm{rank}(\mathbf{A}) = \mathrm{rank}(\mathbf{M})$; unique iff it also equals $n$
- Row reducing $[\mathbf{A}|\mathbf{I}]$ to $[\mathbf{I}|\mathbf{B}]$ gives $\mathbf{A}^{-1} = \mathbf{B}$



# Thank You!

Questions?