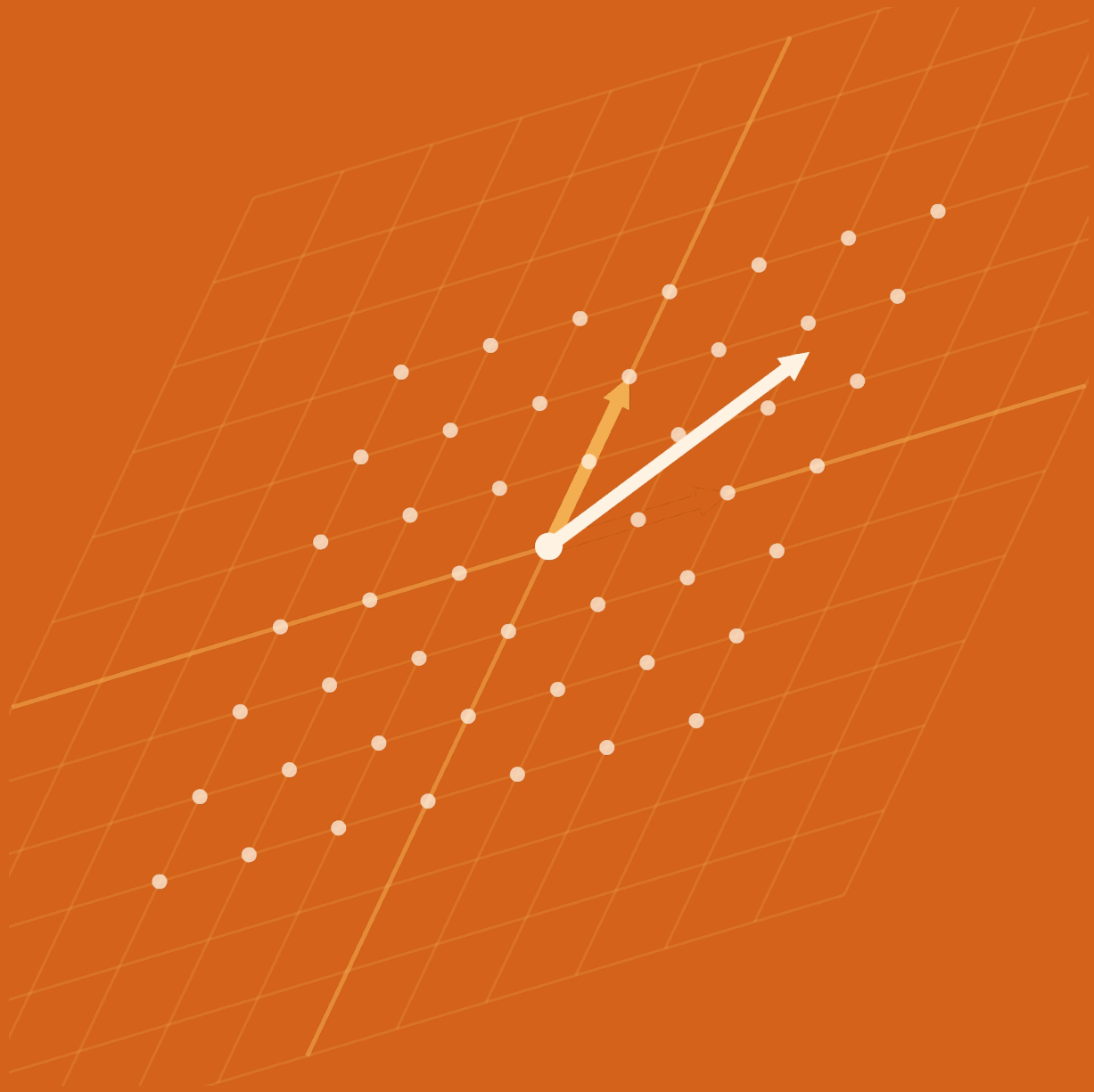
LINEAR ALGEBRA FOR AIML AND 6G

System of Linear Equations: Elementary Operations



Lecture 6

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Outline

1. **Elementary operations**
   Interchange, scaling and combining equations

2. **Gaussian elimination**
   Forward and backward elimination on three examples

01

Elementary operations

Interchange, scaling and combining equations



Elementary operations

- The following operations are called **elementary operations**
  - on a system of linear equations $L_1, L_2, \ldots, L_m$

Interchange

- **Interchange** two of the equations.
- We indicate that the equations L_i and L_j are interchanged by writing

$$L_i \leftrightarrow L_j$$

Multiple

- Replace an equation by a **nonzero multiple** of itself
- We indicate that equation L_i is replaced by kL_i

$$kL_i \rightarrow L_i$$

Combine

- Replace an equation by the **sum of a multiple of another equation and itself**.

- We indicate that equation L_j is replaced by the sum of kL_i and L_j by writing

$$L_j + kL_i \rightarrow L_j$$

02

Gaussian elimination

Forward and backward elimination on three examples



Gaussian elimination

- The main method for solving the general system of linear equations is called **Gaussian elimination**.
  - It essentially consists of two parts:
  - **Forward Elimination**
  - **Backward Elimination**
- These are described next

Forward/ Backward elimination

- **Forward Elimination** Step-by-step reduction of the system yielding an equivalent simpler system

- **Backward Elimination** Step-by-step back-substitution to find the solution of the simpler system.

Ex.1: Unique solution

- Consider the example below

$$L_1 : x - 2y + 3z = 13$$
$$L_2 : 2x - y + 5z = 20$$
$$L_3 : -3x + 4y - 7z = -31$$

- We use the coefficient 1 of x in the first equation L_1 as the **pivot**
  - in order to eliminate x from L_2 and L_3.

Ex.1: Unique solution

$$-2L_1 + L_2 \to L_2$$

- Multiply L_1 by -2 and add to L_2

$$L_1 : x - 2y + 3z = 13$$
$$-2L_1 : -2x + 4y - 6z = -26$$
$$L_2 : 2x - y + 5z = 20$$
$$-2L_1 + L_2 : 3y - z = -6 \to L_2$$

Ex.1: Unique solution

$$3L_1 + L_3 \to L_3$$

- Multiply L_1 by 3 and add to L_3

$$L_1 : x - 2y + 3z = 13$$
$$3L_1 : 3x - 6y + 9z = 39$$
$$L_3 : -3x + 4y - 7z = -31$$
$$3L_1 + L_3 : -2y + 2z = 8 \to L_3$$

Ex.1: Unique solution

- Thus the original system is replaced by

$$L_1 : x - 2y + 3z = 13$$
$$L_2 : 3y - z = -6$$
$$L_3 : -2y + 2z = 8$$

Ex.1: Unique solution

- Now we use the coefficient 3 of y in L_2 as a pivot to eliminate y from L_3

$$\frac{2}{3}L_2 + L_3 \to L_3$$

Gaussian elimination-Ex.1

$$L_1 : x - 2y + 3z = 13$$

$$L_2 : 3y - z = -6$$

$$\frac{2}{3}L_2 : 2y - \frac{2}{3}z = -4$$

$$L_3 : -2y + 2z = 8$$

$$\frac{2}{3}L_2 + L_3 : \frac{4}{3}z = 4 \Rightarrow 4z = 12 \rightarrow L_3$$

Ex.1: Unique solution

- The system is now in **triangular form**

$$L_1 : x - 2y + 3z = 13$$
$$L_2 : 3y - z = -6$$
$$L_3 : 4z = 12$$

Ex.1: Unique solution

- This can be written in the **matrix form**

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 3 & -1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ -6 \\ 12 \end{bmatrix}$$

Ex.1: Unique solution

- The values for the unknowns z, y, x are now solved in **reverse order**

- First solve L_3

$$L_3 : 4z = 12 \Rightarrow z = 3$$

Ex.1: Unique solution

- Now solve L_2 followed by L_1

$$L_2 : 3y - z = -6$$
$$\Rightarrow 3y - 3 = -6 \Rightarrow y = -1$$
$$L_1 : x - 2y + 3z = 13$$
$$\Rightarrow x + 2 + 9 = 13 \Rightarrow x = 2$$

- Thus, this system has a **unique solution**.

Ex.2: Infinitely many soln.

- Consider now the system

$$L_1 : x + 3y - 2z = 2$$
$$L_2 : 2x + 7y - 5z = 5$$
$$L_3 : 3x + 11y - 8z = 8$$

Ex.2: Infinitely many soln.

- Apply now the elementary operations below

$$-2L_1 + L_2 \to L_2, \ -3L_1 + L_3 \to L_3$$

- This leads to the system

$$\begin{aligned} L_1 &: x + 3y - 2z = 2 \\ L_2 &: y - z = 1 \\ L_3 &: 2y - 2z = 2 \end{aligned}$$

  - Note that L_3 is simply a scaled version of L_2

Ex.2: Infinitely many soln.

- Thus, L_3 can be **eliminated**
- This can therefore be reduced to the system

$$L_1 : x + 3y - 2z = 2$$
$$L_2 : y - z = 1$$

  - Note that last equation **does not have a unique solution**

Ex.2: Infinitely many soln.

- To obtain the general solution set $z = a$ in L_2
  - This is termed a **free variable**
- Solving this yields

$$L_2 : y - z = 1$$
$$\Rightarrow y = 1 + a$$

Ex.2: Infinitely many soln.

- Solving L_1 now yields

$$L_1 : x + 3y - 2z = 2$$
$$\Rightarrow x = 2 - 3(1 + a) + 2a$$
$$\Rightarrow x = -1 - a$$

Ex.2: Infinitely many soln.

- Thus a general solution of this system is

$$x = -1 - a, \quad y = 1 + a, \quad z = a$$

- Which yields a valid solution for every value of a
  - Thus, this system has **infinitely many solutions**.

Ex.3: No solution

- Consider the system below

$$L_1 : x_1 + x_2 - x_3 + 3x_4 = 3$$
$$L_2 : 2x_1 + 5x_2 + x_3 + 4x_4 = 8$$
$$L_3 : 3x_1 - 3x_2 - 9x_3 + 13x_4 = 0$$

Ex.3: No solution

- Now perform the operations

$$-2L_1 + L_2 \to L_2,\ -3L_1 + L_3 \to L_3$$

- This leads to the equivalent system

$$\begin{aligned} L_1 &: x_1 + x_2 - x_3 + 3x_4 = 3 \\ L_2 &: 3x_2 + 3x_3 - 2x_4 = 2 \\ L_3 &: -6x_2 - 6x_3 + 4x_4 = -9 \end{aligned}$$

Ex.3: No solution

- Perform now the operation

$$2L_2 + L_3 \rightarrow L_3$$

- This further leads to

$$L_1 : x_1 + x_2 - x_3 + 3x_4 = 3$$
$$L_2 : 3x_2 + 3x_3 - 2x_4 = 2$$
$$L_3 : 0x_2 + 0x_3 + 0x_4 = -5$$

Ex.3: No solution

- Note that last equation is of the form

$$0x_2 + 0x_3 + 0x_4 = -5$$

- System is **inconsistent**
  - Hence, this system has **no solution**

Gaussian elimination

- Thus, **Gaussian elimination** can be used to reduce a system of equations

  - and further determine if it has a **unique solution**, **infinite solutions** or **no solution**

Summary

- Elementary operations: $L_i \leftrightarrow L_j$, $kL_i \to L_i$, $L_j + kL_i \to L_j$
- Elementary operations give an equivalent system
- Gaussian elimination: forward elimination, then back-substitution
- A triangular system with nonzero pivots has a unique solution
- A redundant equation leaves a free variable: infinitely many solutions
- A row $0 = c$ with $c \neq 0$ means the system is inconsistent



Thank You!

Questions?