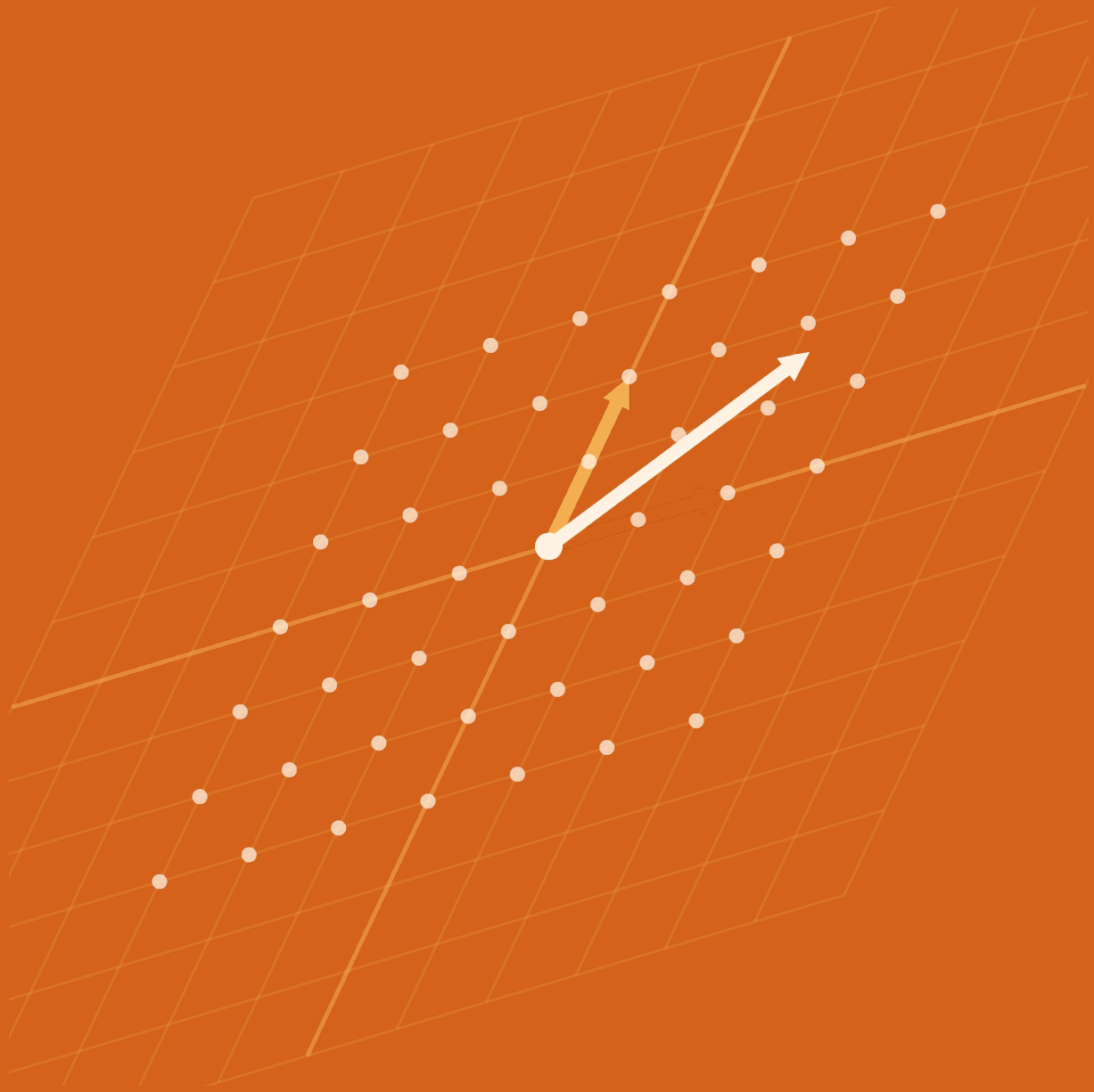
LINEAR ALGEBRA FOR AIML AND 6G

Matrices: Inverse and Orthogonality



Lecture 4

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Outline

1. **Matrix inverse**
   Invertible matrices, uniqueness and the 2 × 2 inverse formula

2. **Diagonal and Triangular matrices**
   Diagonal, upper triangular and lower triangular matrices

3. **Special matrices**
   Symmetric, skew-symmetric and orthogonal matrices

4. **Complex matrices**
   Hermitian, skew-Hermitian and unitary matrices

01

Matrix inverse

Invertible matrices, uniqueness and the 2 × 2 inverse formula



Invertible matrix

- Consider a **square matrix** $\mathbf{A}$
- This is termed **invertible** or **non-singular**
  - if there exists another matrix $\mathbf{B}$ such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}$$

Inverse - example

- Consider $\mathbf{A}$ below

$$\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$$

- Its inverse is

$$\mathbf{A}^{-1} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$$

Inverse - example

- It can be seen

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Similarly

$$\mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Inverse - properties

- If inverse exists, it is unique

$$\mathbf{AB}_1 = \mathbf{I} = \mathbf{AB}_2 \Leftrightarrow \mathbf{B}_1 = \mathbf{B}_2$$

Inverse - properties

- Consider two **invertible matrices** $\mathbf{A}, \mathbf{B}$

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Inverse matrix

- Consider a 2×2 matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- Its **inverse** is given as

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Inverse - Example

- Consider a 2×2 matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$$

- Its inverse is given as

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{1} \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$$

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Diagonal and Triangular matrices

Diagonal, upper triangular and lower triangular matrices



Diagonal matrix

- A matrix $\mathbf{A}$ is **diagonal** if all its non-diagonal elements are zero

$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

Diagonal matrix - example

- Example of a 3×3 diagonal matrix is below

$$\mathbf{A} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Diagonal matrix

- Such a matrix is sometimes represented as

$$\mathbf{D} = \text{diag}\left(d_{11}, d_{22}, \ldots, d_{nn}\right)$$

Upper triangular matrix

- Square matrix is **upper triangular** if all entries below diagonal are zero

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix}$$

Lower triangular matrix

- Square matrix is **lower triangular** if all entries above diagonal are zero

$$\mathbf{B} = \begin{bmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

03

Special matrices

Symmetric, skew-symmetric and orthogonal matrices



Symmetric matrix

- Square matrix is **symmetric** if

$$\mathbf{A}^T = \mathbf{A}$$

Symmetric matrix -example

- Example of **symmetric matrix**

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & -6 \\ 1 & -2 & 3 \\ -6 & 3 & 7 \end{bmatrix}$$

Skew-symmetric matrix

- Square matrix is **skew-symmetric** if

$$\mathbf{A}^T = -\mathbf{A}$$

Skew-symmetric-example

- Example of skew-symmetric matrix

$$\mathbf{B} = \begin{bmatrix} 0 & -2 & 6 \\ 2 & 0 & -1 \\ -6 & 1 & 0 \end{bmatrix}$$

Orthogonal matrix

- A real matrix is orthogonal if $\mathbf{A}^T = \mathbf{A}^{-1}$

$$\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T = \mathbf{I}$$

Orthogonal matrix example

- Matrix below is orthogonal

$$\mathbf{A} = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

$$\mathbf{A}\mathbf{A}^T = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}\frac{1}{5}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Orthonormal set

- Consider **orthogonal matrix** $\mathbf{A}$ with columns $\bar{\mathbf{a}}_i$
- Columns of $\mathbf{A}$ form an orthonormal set

$$\|\bar{\mathbf{a}}_i\| = 1$$

$$\bar{\mathbf{a}}_i^T \bar{\mathbf{a}}_j = 0 \text{ for } i \neq j$$

- Similarly rows also form an **orthonormal set**

Orthonormal set - example

- Consider the **orthogonal matrix** $\mathbf{A}$ below

$$\mathbf{A} = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

Orthonormal set - example

- The columns of $\mathbf{A}$ are

$$\bar{\mathbf{a}}_1 = \frac{1}{5}\begin{bmatrix}3\\4\end{bmatrix}, \quad \bar{\mathbf{a}}_2 = \frac{1}{5}\begin{bmatrix}4\\-3\end{bmatrix}$$

- Observe that

$$\|\bar{\mathbf{a}}_1\| = \|\bar{\mathbf{a}}_2\| = 1$$
$$\bar{\mathbf{a}}_1^T \bar{\mathbf{a}}_2 = 0$$

Orthogonal matrix

- For any value of θ, matrices below are **orthogonal**

$$\mathbf{A} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

Orthogonal matrix

- Consider $\theta = \pi/6$, matrix below is orthogonal

$$\mathbf{A} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix}$$

Orthogonal matrix

- Consider $\theta = \pi/6$, matrix below is orthogonal

$$\mathbf{B} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{bmatrix}$$

04

Complex matrices

Hermitian, skew-Hermitian and unitary matrices



Complex matrices

- A **complex matrix** $\mathbf{A}$ has complex elements
- An example is given below

$$\mathbf{A} = \begin{bmatrix} 3-2i & 1+5i & 7i \\ 4+i & 2-6i & 5-3i \end{bmatrix}$$

Hermitian transpose

- **Hermitian transpose** of complex matrix is obtained by
  - **transpose** and **complex conjugate** of each element

Hermitian transpose-Ex.

$$\mathbf{A} = \begin{bmatrix} 3-2i & 1+5i & 7i \\ 4+i & 2-6i & 5-3i \end{bmatrix}$$

- The **Hermitian transpose** of the matrix above is

$$\mathbf{A}^H = \begin{bmatrix} 3+2i & 4-i \\ 1-5i & 2+6i \\ -7i & 5+3i \end{bmatrix}$$

Hermitian matrix

- Matrix $\mathbf{A}$ is **Hermitian** if

$$\mathbf{A} = \mathbf{A}^H$$

Hermitian matrix-Ex.

- Matrix $\mathbf{A}$ below is **Hermitian**

$$\mathbf{A} = \begin{bmatrix} 5 & 3-2i \\ 3+2i & -1 \end{bmatrix} = \mathbf{A}^H$$

Skew-Hermitian matrix

- Matrix $\mathbf{A}$ is **Skew-Hermitian** if

$$\mathbf{A} = -\mathbf{A}^H$$

Skew-Hermitian – Ex.

- Example of a **Skew-Hermitian** matrix is

$$\mathbf{A} = \begin{bmatrix} 0 & 2+3i \\ -2+3i & 0 \end{bmatrix}$$

$$\mathbf{A}^H = \begin{bmatrix} 0 & -2-3i \\ 2-3i & 0 \end{bmatrix} = -\mathbf{A}$$

Unitary matrix

- Matrix $\mathbf{A}$ is **unitary** if

$$\mathbf{A}^{-1} = \mathbf{A}^{H}$$
$$\mathbf{A}\mathbf{A}^{H} = \mathbf{A}^{H}\mathbf{A} = \mathbf{I}$$

Unitary matrix - example

- Example of a **unitary matrix** is below

$$\mathbf{A} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

$$\mathbf{A}^H \mathbf{A} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Orthonormal set

- A matrix $\mathbf{A}$ is unitary if and only if its columns form an **orthonormal set**

$$\|\bar{\mathbf{a}}_i\| = 1$$

$$\bar{\mathbf{a}}_i^H \bar{\mathbf{a}}_j = 0 \text{ for } i \neq j$$

Orthonormal set – Ex.

- Consider the unitary matrix below

$$\mathbf{A} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

- Its columns are

$$\bar{\mathbf{a}}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}, \quad \bar{\mathbf{a}}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Orthonormal set – Ex.

- Observe now that

$$\|\bar{\mathbf{a}}_1\|^2 = \frac{1}{2}(|1|^2 + |i|^2)$$

$$= \frac{1}{2}(1+1) = 1 = \|\bar{\mathbf{a}}_2\|^2$$

$$\bar{\mathbf{a}}_1^H \bar{\mathbf{a}}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}i \\ 1\end{bmatrix}$$

$$= \frac{1}{2}(i - i) = 0$$

Summary

- $\mathbf{A}$ is invertible if $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$; the inverse is unique
- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$; 2 × 2 inverse uses $1/(ad - bc)$
- Diagonal and triangular matrices have zeros off / below / above the diagonal
- Symmetric: $\mathbf{A}^T = \mathbf{A}$; skew-symmetric: $\mathbf{A}^T = -\mathbf{A}$
- Orthogonal: $\mathbf{A}^T\mathbf{A} = \mathbf{I}$; columns form an orthonormal set
- Hermitian: $\mathbf{A} = \mathbf{A}^H$; unitary: $\mathbf{A}^H\mathbf{A} = \mathbf{I}$



Thank You!

Questions?