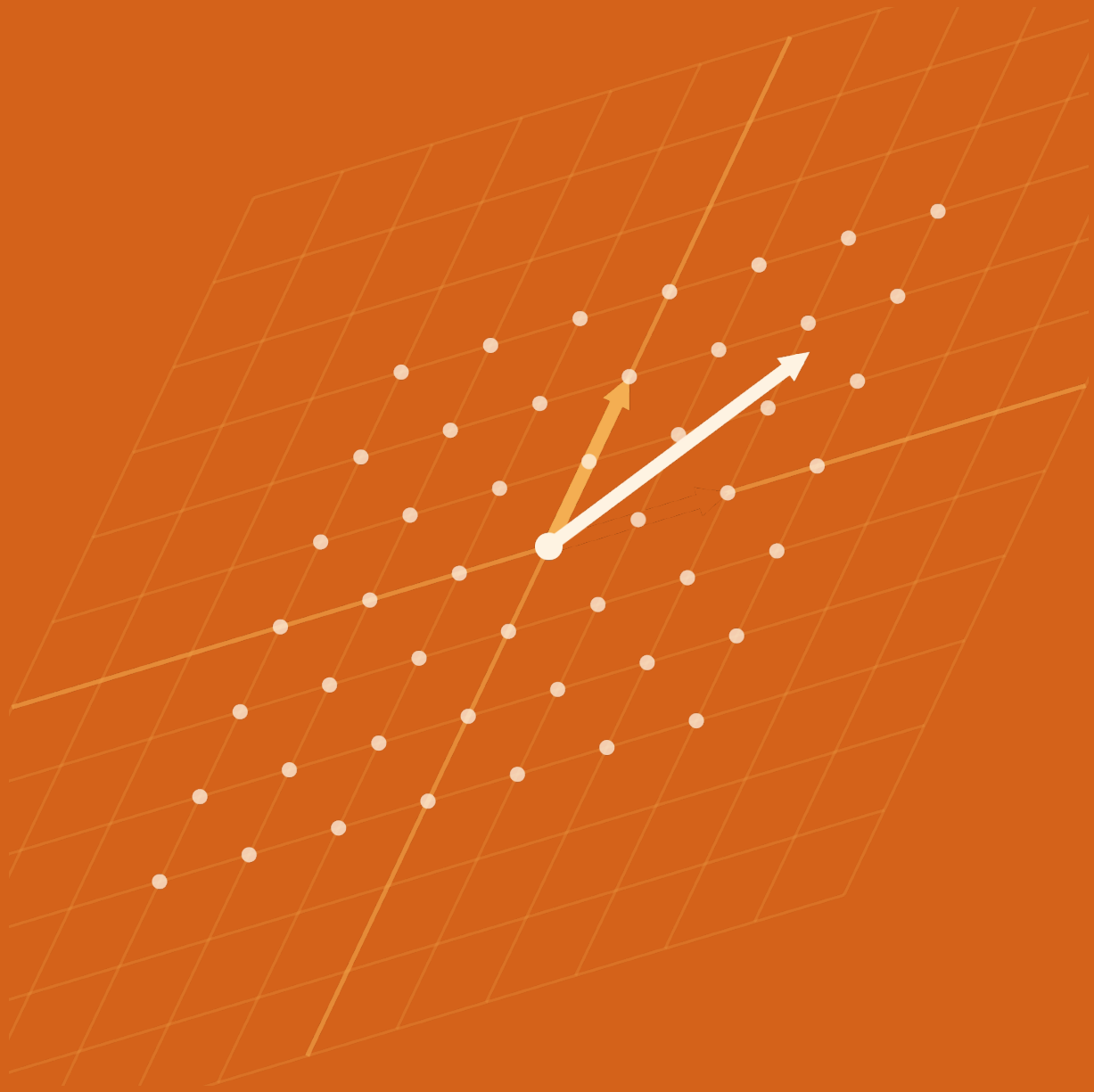
LINEAR ALGEBRA FOR AIML AND 6G

Matrices: Basics and Properties



Lecture 3

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Outline

1. **Matrix Basics**
   Size, rows, columns, elements and special vectors

2. **Matrix operations**
   Addition, scalar multiplication, products and transpose

3. **Square matrices**
   Diagonal, trace and the identity matrix

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Matrix Basics

Size, rows, columns, elements and special vectors



Matrix

- A **matrix** is a rectangular array of scalars

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Matrix size

- The **size** of the matrix is $m \times n$
- m is number of rows
- n is number of columns

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Matrix - example

- Example of a 2×3 matrix is below

$$\mathbf{A} = \begin{bmatrix} 4 & -1 & 3 \\ -2 & 6 & -7 \end{bmatrix}$$

- **Number of rows** $= 2$
- **Number of columns** $= 3$

Matrix rows

- Rows of the matrix are

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \end{bmatrix}$$
$$\begin{bmatrix} a_{21} & a_{22} & \dots & a_{2n} \end{bmatrix}$$
$$\vdots$$
$$\begin{bmatrix} a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Matrix rows - example

$$\mathbf{A} = \begin{bmatrix} 4 & -1 & 3 \\ -2 & 6 & -7 \end{bmatrix}$$

- Rows of the matrix are

$$\begin{bmatrix} 4 & -1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 6 & -7 \end{bmatrix}$$

Matrix columns

- Columns of the matrix are

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Matrix columns - example

$$\mathbf{A} = \begin{bmatrix} 4 & -1 & 3 \\ -2 & 6 & -7 \end{bmatrix}$$

- Columns of the matrix are

$$\begin{bmatrix} 4 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ -7 \end{bmatrix}$$

Matrix elements

- a_{ij} is the element in i-th row and j-th column

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Matrix elements - example

$$\mathbf{A} = \begin{bmatrix} 4 & -1 & 3 \\ -2 & 6 & -7 \end{bmatrix}$$

- In the above matrix $a_{21} = -2$, $a_{13} = 3$

Row vector

- A matrix with only one row is called a **row matrix** **or** **row vector**
  - It has size $1 \times n$
- Example of a **row vector** is below.

$$\begin{bmatrix} 3 & -7 & 1 & 5 \end{bmatrix}$$

Column vectors

- A matrix with only one column is called a **column matrix or column vector**.
  - It has size $m \times 1$
- An example of a **column vector** is

$$\bar{\mathbf{u}} = \begin{bmatrix} -4 \\ 2 \\ 5 \\ -1 \end{bmatrix}$$

Zero matrix

- A matrix whose entries are all zero is called a **zero matrix**

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

  - And will usually be denoted by 0.

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Matrix operations

Addition, scalar multiplication, products and transpose



Matrix addition

- Let $\mathbf{A}$ and $\mathbf{B}$ be two matrices of the same size.

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{bmatrix}$$

Scalar multiplication

- Let $\mathbf{A}$ be a matrix

$$k\mathbf{A} = \begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{bmatrix}$$

Negative of matrix

- **Negative of the matrix** is given as below

$$(-1)\,\mathbf{A} = -\mathbf{A}$$

$$-\mathbf{A} = \begin{bmatrix} -a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & -a_{22} & \dots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{m1} & -a_{m2} & \dots & -a_{mn} \end{bmatrix}$$

Matrix subtraction

- **Matrix subtraction** naturally follows as below

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$$

$$\mathbf{A} - \mathbf{B}$$

$$= \begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} & \dots & a_{1n} - b_{1n} \\ a_{21} - b_{21} & a_{22} - b_{22} & \dots & a_{2n} - b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} - b_{m1} & a_{m2} - b_{m2} & \dots & a_{mn} - b_{mn} \end{bmatrix}$$

Other properties-addition

- Other properties related to **addition** are below

$$(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$$

$$\mathbf{A}+\mathbf{0}=\mathbf{0}+\mathbf{A}=\mathbf{A}$$

$$\mathbf{A}+(-\mathbf{A})=(-\mathbf{A})+\mathbf{A}=\mathbf{0}$$

$$\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$$

Other prop. – scalar mult.

- Other properties related to **scalar multiplication** are below

$$k\left(\mathbf{A}+\mathbf{B}\right)=k\mathbf{A}+k\mathbf{B}$$

$$\left(k+k'\right)\mathbf{A}=k\mathbf{A}+k'\mathbf{A}$$

$$\left(kk'\right)\mathbf{A}=k\left(k'\mathbf{A}\right)$$

$$1\mathbf{A}=\mathbf{A}$$

Matrix multiplication

- **Matrix multiplication** is defined as follows

- Suppose $\mathbf{A}$ and $\mathbf{B}$ are matrices such that the **number of columns** of $\mathbf{A}$ is equal to the **number of rows** of $\mathbf{B}$

Matrix multiplication

- Suppose $\mathbf{A}$ and $\mathbf{B}$ are matrices
- Matrix multiplication is defined only when
  - the **number of columns** of $\mathbf{A}$ is equal to the **number of rows** of $\mathbf{B}$

Matrix multiplication

- Say, $\mathbf{A}$ is an $m \times p$ matrix and $\mathbf{B}$ is a $p \times n$ matrix.

- The product $\mathbf{AB}$ is the $m \times n$ matrix whose ij-**entry** is obtained by multiplying the i-**th row** of $\mathbf{A}$ by the j-**th column** of **B**

Matrix multiplication

- This can be seen below

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ip}b_{pj}$$

$$= \sum_{k=1}^{p} a_{ik}b_{kj}$$

$$\mathbf{C} = \begin{bmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ip} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mp} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{p1} & \cdots & b_{pj} & \cdots & b_{pn} \end{bmatrix}$$

row i of $\mathbf{A}$ column j of $\mathbf{B}$

Matrix mult. – example1

- Consider the matrices $\mathbf{A}$, $\mathbf{B}$, defined as below

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 3 & -2 & 1 \\ 4 & 0 & -5 \end{bmatrix}$$

- Note $\mathbf{A} \sim 2 \times 2$ and $\mathbf{B} \sim 2 \times 3$
  - Therefore, number of columns of $\mathbf{A}$ equals number of rows of $\mathbf{B}$, i.e. 2

Matrix mult. – example1

- The **matrix product** $\mathbf{AB}$ is computed as

$$\mathbf{AB} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -2 & 1 \\ 4 & 0 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 6+4 & -4+0 & 2-5 \\ -3+12 & 2+0 & -1-15 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & -4 & -3 \\ 9 & 2 & -16 \end{bmatrix}$$

Matrix mult. – example2

- Consider another example below

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 4 & -1 \\ 2 & 5 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 10 & 3 \\ 10 & 14 \end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix} 4 & -1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 9 & 17 \end{bmatrix}$$

Matrix multiplication

- In general

$$AB \neq BA$$

- Matrix multiplication is **NOT commutative**

Multiplication - properties

- Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be three matrices.

- When the products and sums are defined, following properties hold

$$(\mathbf{AB})\,\mathbf{C} = \mathbf{A}\,(\mathbf{BC})$$

$$\mathbf{A}\,(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$$

$$(\mathbf{B}+\mathbf{C})\,\mathbf{A} = \mathbf{BA}+\mathbf{CA}$$

$$k\,(\mathbf{AB}) = (k\mathbf{A})\,\mathbf{B} = \mathbf{A}\,(k\mathbf{B})$$

$$\mathbf{0A} = \mathbf{A0} = \mathbf{0}$$

Matrix transpose

- **Transpose** of matrix $\mathbf{A}$ is denoted by $\mathbf{A}^T$.
- Columns of $\mathbf{A}$ become rows of $\mathbf{A}^T$.

$$\left[\mathbf{A}^T\right]_{ij} = [\mathbf{A}]_{ji}$$

Matrix transpose-example

- Consider example below

- **Transpose** of row vector is a column vector and vice versa

$$\begin{bmatrix} 3 & -1 & 2 \\ 0 & 4 & 5 \end{bmatrix}^T = \begin{bmatrix} 3 & 0 \\ -1 & 4 \\ 2 & 5 \end{bmatrix}$$

Transpose properties

- **Matrix transpose** obeys the following properties

$$(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$$

$$(k\mathbf{A})^T = k\mathbf{A}^T$$

$$\left(\mathbf{A}^T\right)^T = \mathbf{A}$$

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$$

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Square matrices

Diagonal, trace and the identity matrix



Square matrices

- A **square matrix** has same number of rows as columns
- Size $n \times n$
- **Number of rows = Number of columns =** n

Square matrix - example

- Example of 3×3 square matrix is below

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & -2 \\ 4 & -5 & 0 \\ 2 & 6 & 1 \end{bmatrix}$$

Diagonal

- Let $\mathbf{A}$ be a square matrix of size $n \times n$
- The **diagonal elements** are

$$a_{11}, a_{22}, \ldots, a_{nn}$$

Diagonal - example

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & -2 \\ 4 & -5 & 0 \\ 2 & 6 & 1 \end{bmatrix}$$

- Consider the square matrix $\mathbf{A}$ above
- The diagonal elements are

$$3,\ -5,\ 1$$

Trace

- The **trace** of $\mathbf{A}$ is sum of the diagonal elements

$$\mathrm{tr}\left(\mathbf{A}\right) = a_{11} + a_{22} + \ldots + a_{nn}$$

Trace - example

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & -2 \\ 4 & -5 & 0 \\ 2 & 6 & 1 \end{bmatrix}$$

- Consider the square matrix $\mathbf{A}$ above
- The trace is

$$3 - 5 + 1 = -1$$

Trace properties

- Trace of a matrix exhibits the following properties

$$\mathrm{tr}\left(\mathbf{A}+\mathbf{B}\right)=\mathrm{tr}\left(\mathbf{A}\right)+\mathrm{tr}\left(\mathbf{B}\right)$$

$$\mathrm{tr}\left(\mathbf{A}^{T}\right)=\mathrm{tr}\left(\mathbf{A}\right)$$

$$\mathrm{tr}\left(k\mathbf{A}\right)=k\cdot\mathrm{tr}\left(\mathbf{A}\right)$$

$$\mathrm{tr}\left(\mathbf{AB}\right)=\mathrm{tr}\left(\mathbf{BA}\right)$$

Identity matrix

- Identity matrix of size $n \times n$ is denoted by $\mathbf{I}_n$
- All its **diagonal elements** are equal to 1
  - and **non-diagonal elements** equal to 0

Identity matrix - example

- 3×3 identity matrix is below

$$\mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Identity matrix - properties

- Note that if $\mathbf{A}$ is $n \times n$ matrix

$$\mathbf{A}\mathbf{I}_n = \mathbf{I}_n\mathbf{A} = \mathbf{A}$$

- Note that if $\mathbf{A}$ is $m \times n$ matrix

$$\mathbf{A}\mathbf{I}_n = \mathbf{I}_m\mathbf{A} = \mathbf{A}$$

Summary

- An $m \times n$ matrix has m rows and n columns; a_{ij} is the entry in row i, column j
- Matrices of the same size add entry by entry; a scalar multiplies every entry
- $\mathbf{AB}$ is defined when columns of $\mathbf{A}$ = rows of $\mathbf{B}$: $c_{ij} = \sum_k a_{ik} b_{kj}$
- In general $\mathbf{AB} \neq \mathbf{BA}$: matrix multiplication is not commutative

Summary

- Transpose swaps rows and columns; $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$
- Trace is the sum of diagonal entries; $\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$
- Identity: $\mathbf{AI}_n = \mathbf{I}_m\mathbf{A} = \mathbf{A}$



Thank You!

Questions?