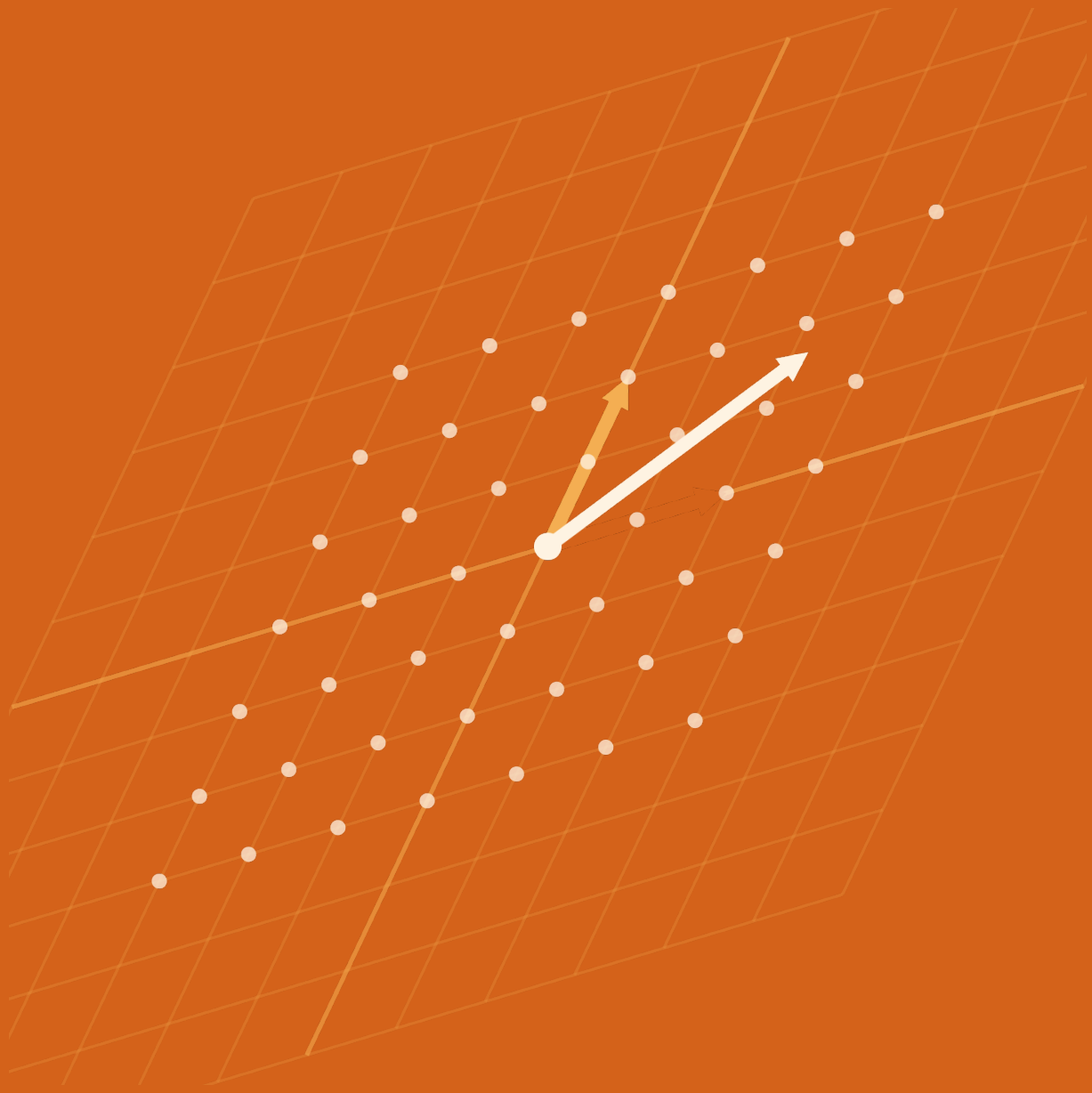
LINEAR ALGEBRA FOR AIML AND 6G

Vectors: Norm and Projection



Lecture 2

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Outline

1. **Norm**
   Length of a vector, unit vectors and the norm properties

2. **Angle and Projection**
   Angle between two vectors and projection of one on another

3. **Complex vectors**
   Vectors with complex entries, Hermitian transpose and norm

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Norm

Length of a vector, unit vectors and the norm properties



Norm of vector

- The **norm** of a vector $\bar{\mathbf{u}}$ is defined as

$$\|\bar{\mathbf{u}}\| = \sqrt{\bar{\mathbf{u}} \cdot \bar{\mathbf{u}}} = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$$

$$\|\bar{\mathbf{u}}\|^2 = u_1^2 + u_2^2 + \ldots + u_n^2$$

Norm of vector - example

- Consider the vector

$$\bar{\mathbf{u}} = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 4 \\ -1 \end{bmatrix}$$

Norm of vector - example

- Its **norm** is calculated as

$$\|\bar{\mathbf{u}}\| = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$$

$$= \sqrt{4 + 9 + 1 + 16 + 1} = \sqrt{31}$$

$$\|\bar{\mathbf{u}}\|^2 = u_1^2 + u_2^2 + \ldots + u_n^2 = 31$$

Unit vector

- A unit vector satisfies

$$\|\bar{\mathbf{u}}\| = 1$$

Unit vector

- For any non-zero vector $\bar{\mathbf{v}}$ below is a **unit vector** in the **same direction**

$$\frac{\bar{\mathbf{v}}}{\|\bar{\mathbf{v}}\|}$$

- This is also called **normalizing** $\bar{\mathbf{v}}$

Unit vector - example

- Consider again the vector

$$\bar{\mathbf{u}} = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 4 \\ -1 \end{bmatrix}$$

$$\|\bar{\mathbf{u}}\| = \sqrt{31}$$

Unit vector - example

- **Unit vector** is

$$\frac{\bar{\mathbf{u}}}{\|\bar{\mathbf{u}}\|} = \frac{1}{\sqrt{31}} \begin{bmatrix} 2 \\ -3 \\ 1 \\ 4 \\ -1 \end{bmatrix}$$

Properties of norm

- **Non-negativity** and **positive definiteness**

$$\|\bar{\mathbf{u}}\| \geq 0$$

$$\|\bar{\mathbf{u}}\| = 0 \Leftrightarrow \bar{\mathbf{u}} = \mathbf{0}$$

Properties of norm

- **Homogeneity**

- For any scalar k

$$\|k\bar{\mathbf{u}}\| = |k| \, \|\bar{\mathbf{u}}\|$$

Properties of norm

- **Triangle inequality** (or subadditivity)

$$\|\bar{\mathbf{v}} + \bar{\mathbf{w}}\| \leq \|\bar{\mathbf{v}}\| + \|\bar{\mathbf{w}}\|$$

Cauchy-Schwarz Ineq.

- **Cauchy-Schwarz Inequality** states for two vectors $\bar{\mathbf{u}}$, $\bar{\mathbf{v}}$, we have

$$\left|\bar{\mathbf{u}} \cdot \bar{\mathbf{v}}\right| \leq \left\|\bar{\mathbf{u}}\right\| \left\|\bar{\mathbf{v}}\right\|$$

Cauchy-Schwarz Ineq.

- This implies

$$-\|\bar{\mathbf{u}}\|\,\|\bar{\mathbf{v}}\| \leq \bar{\mathbf{u}}\cdot\bar{\mathbf{v}} \leq \|\bar{\mathbf{u}}\|\,\|\bar{\mathbf{v}}\|$$

$$\Rightarrow -1 \leq \frac{\bar{\mathbf{u}}\cdot\bar{\mathbf{v}}}{\|\bar{\mathbf{u}}\|\,\|\bar{\mathbf{v}}\|} \leq 1$$

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Angle and Projection

Angle between two vectors and projection of one on another



Angle between vectors

- Angle θ between the vectors is defined as

$$\cos\,\theta = \frac{\bar{\mathbf{u}} \cdot \bar{\mathbf{v}}}{\|\bar{\mathbf{u}}\|\,\|\bar{\mathbf{v}}\|}$$

- Naturally, if $\bar{\mathbf{u}} \perp \bar{\mathbf{v}}$, then $\bar{\mathbf{u}} \cdot \bar{\mathbf{v}} = 0$, which implies $\cos\,\theta = 0$

Angle – Example

- Consider the vectors

$$\bar{\mathbf{u}} = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}, \bar{\mathbf{v}} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$\bar{\mathbf{u}} \cdot \bar{\mathbf{v}} = 3 - 4 + 8 = 7$$

Angle – Example

- The **angle** between the vectors is

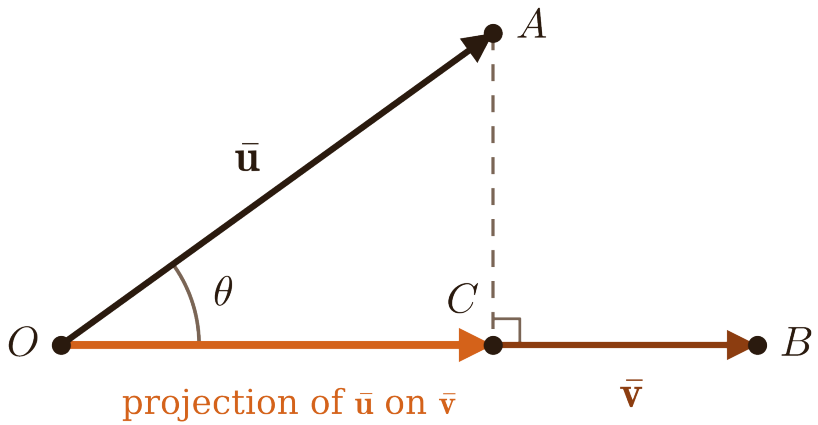
$$\|\bar{\mathbf{u}}\| = \sqrt{17}, \|\bar{\mathbf{v}}\| = \sqrt{21}$$

$$\cos\theta = \frac{\bar{\mathbf{u}} \cdot \bar{\mathbf{v}}}{\|\bar{\mathbf{u}}\| \, \|\bar{\mathbf{v}}\|} = \frac{7}{\sqrt{17} \times \sqrt{21}}$$

Projection of vector

- **Projection** of vector $\bar{\mathbf{u}}$, on $\bar{\mathbf{v}}$ is given as

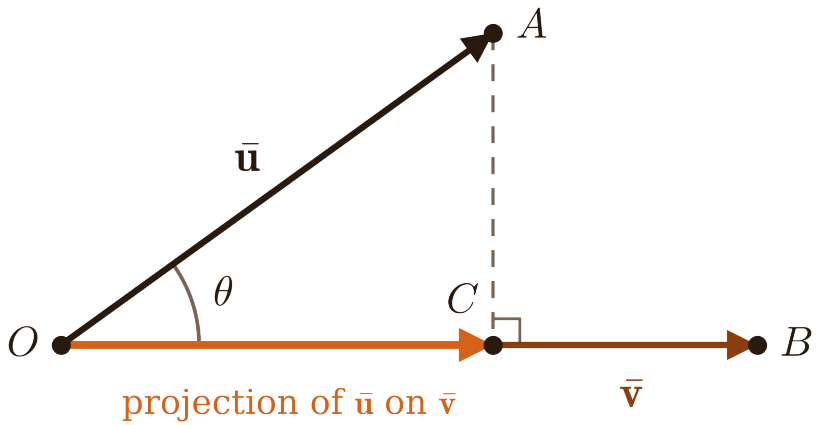
$$\left(\bar{\mathbf{u}} \cdot \frac{\bar{\mathbf{v}}}{\|\bar{\mathbf{v}}\|}\right) \frac{\bar{\mathbf{v}}}{\|\bar{\mathbf{v}}\|} = \frac{\left(\bar{\mathbf{u}} \cdot \bar{\mathbf{v}}\right) \bar{\mathbf{v}}}{\|\bar{\mathbf{v}}\|^2}$$



Projection - example

- Consider

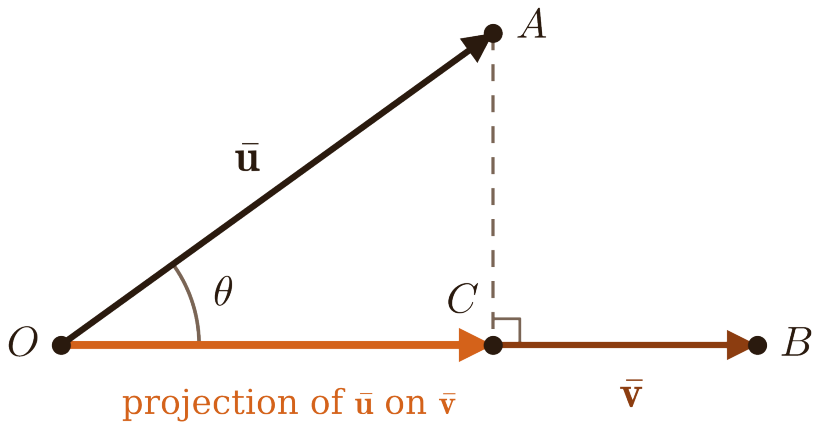
$$\bar{\mathbf{u}} = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}, \bar{\mathbf{v}} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$



Projection - example

- **Projection** of vector $\bar{\mathbf{u}}$, on $\bar{\mathbf{v}}$ is given as

$$\frac{(\bar{\mathbf{u}} \cdot \bar{\mathbf{v}})\,\bar{\mathbf{v}}}{\|\bar{\mathbf{v}}\|^2} = \left(\frac{1}{\sqrt{21}}\right)^2 \times 7 \times \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$



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Complex vectors

Vectors with complex entries, Hermitian transpose and norm



Complex vectors

- **Complex vector** $\bar{\mathbf{u}}$ has complex numbers as elements

$$\bar{\mathbf{u}} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

Complex - example

- Example of a **complex vector** is below

$$\bar{\mathbf{u}} = \begin{bmatrix} 5+i \\ 2-3i \\ 1+3i \end{bmatrix}$$

Hermitian transpose

- Hermitian transpose of a vector is defined as the **conjugate transpose**

$$\bar{\mathbf{u}}^H = \begin{bmatrix} u_1^* & u_2^* & \dots & u_n^* \end{bmatrix}$$

Hermitian - example

- Example of **Hermitian transpose** of a vector is below

$$\bar{\mathbf{u}} = \begin{bmatrix} 5+i \\ 2-3i \\ 1+3i \end{bmatrix}$$

$$\bar{\mathbf{u}}^H = \begin{bmatrix} 5-i & 2+3i & 1-3i \end{bmatrix}$$

Inner product

- **Inner product** between complex vectors is defined as

$$\bar{\mathbf{u}}^H \bar{\mathbf{v}} = \begin{bmatrix} u_1^* & u_2^* & \dots & u_n^* \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$= u_1^* v_1 + u_2^* v_2 + \ldots + u_n^* v_n$$

Norm

- The **norm** of a complex vector is

$$\|\bar{\mathbf{u}}\| = \sqrt{\bar{\mathbf{u}}^H \bar{\mathbf{u}}}$$

$$= \sqrt{u_1^* u_1 + u_2^* u_2 + \ldots + u_n^* u_n}$$

$$= \sqrt{|u_1|^2 + |u_2|^2 + \ldots + |u_n|^2}$$

Norm - example

- Consider

$$\bar{\mathbf{u}} = \begin{bmatrix} 5+i \\ 2-3i \\ 1+3i \end{bmatrix}$$

$$\|\bar{\mathbf{u}}\| = \sqrt{5^2 + 1^2 + 2^2 + 3^2 + 1^2 + 3^2}$$

$$= \sqrt{49} = 7$$

Summary

- Norm: $\|\bar{\mathbf{u}}\| = \sqrt{(\bar{\mathbf{u}} \cdot \bar{\mathbf{u}})}$, the length of the vector
- Unit vector in the direction of $\bar{\mathbf{v}}$ is $\bar{\mathbf{v}}/\|\bar{\mathbf{v}}\|$ (normalizing)
- Norm is non-negative, homogeneous and obeys the triangle inequality
- Cauchy-Schwarz: $|\bar{\mathbf{u}} \cdot \bar{\mathbf{v}}| \leq \|\bar{\mathbf{u}}\|\|\bar{\mathbf{v}}\|$
- Angle: $\cos\theta = \bar{\mathbf{u}} \cdot \bar{\mathbf{v}}/\|\bar{\mathbf{u}}\|\|\bar{\mathbf{v}}\|$; projection of $\bar{\mathbf{u}}$ on $\bar{\mathbf{v}}$ is $(\bar{\mathbf{u}} \cdot \bar{\mathbf{v}})\bar{\mathbf{v}}/(\|\bar{\mathbf{v}}\|^2)$
- Complex vectors: inner product $\bar{\mathbf{u}}^H\bar{\mathbf{v}}$ and norm $\|\bar{\mathbf{u}}\| = \sqrt{(\bar{\mathbf{u}}^H\bar{\mathbf{u}})}$



Thank You!

Questions?