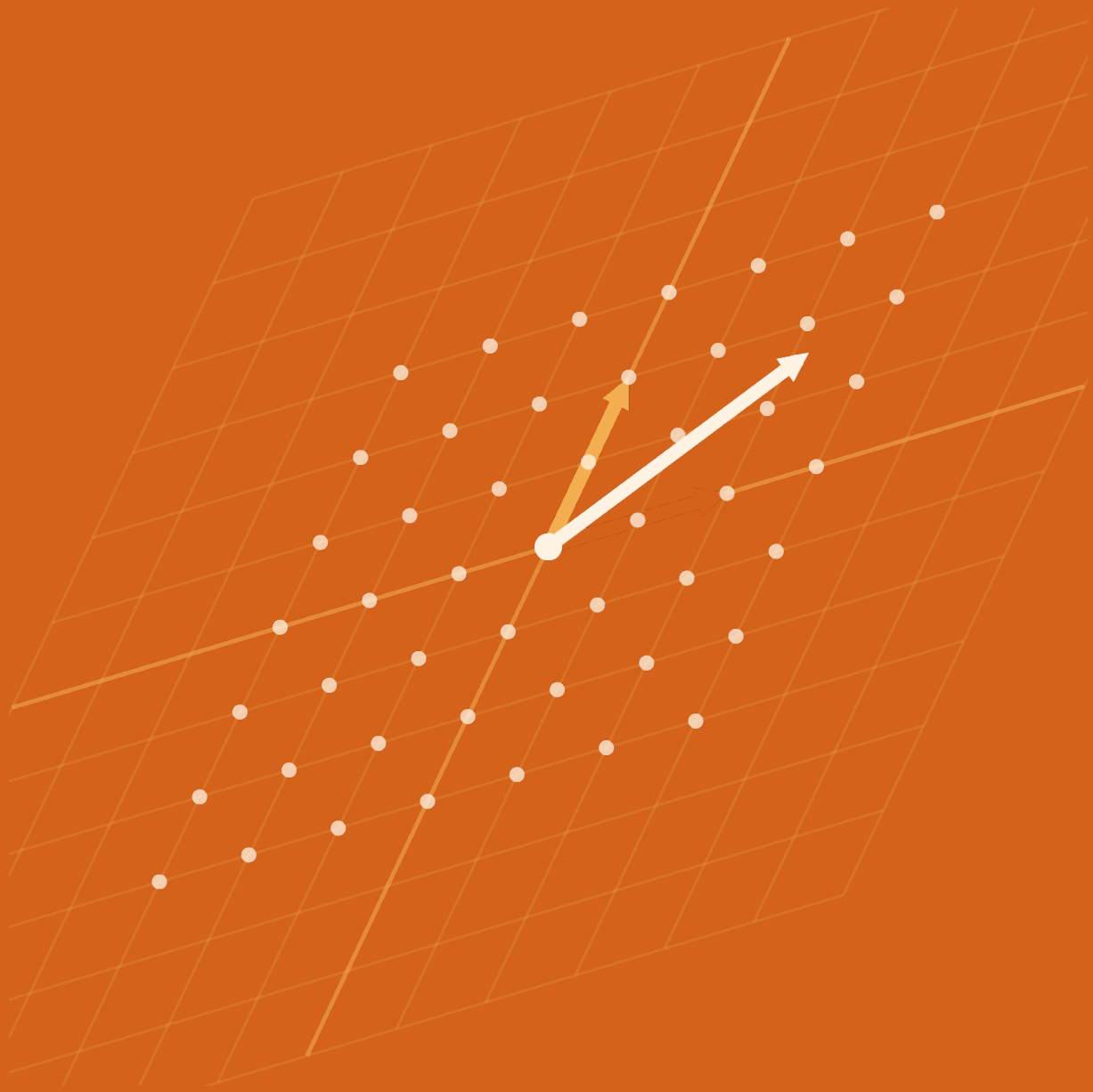
LINEAR ALGEBRA FOR AIML AND 6G

Vectors: Basic Operations and Properties

Lecture 1

Prof. Aditya K. Jagannatham
IIT Kanpur



Outline

1. **Definition and Basic Operations**
   Vectors in $\mathbb{R}^n$, transpose, sum, scalar product, linear combination
2. **Vectors in Physics**
   Vectors as arrows: origin, addition and scaling
3. **Properties of Vectors**
   Rules for vector addition and scalar multiplication
4. **Dot Product**
   Inner product, orthogonality and its properties

01

Definition and Basic Operations

Vectors in $\mathbb{R}^n$, transpose, sum, scalar product, linear combination



Vectors in $\mathbb{R}^n$

- An n-tuple of real numbers is termed a **vector**.
- It is denoted as

$$\bar{\mathbf{u}} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

Example

- An example of an $n = 4$ dimensional vector is below.

$$\bar{\mathbf{u}} = \begin{bmatrix} 5 \\ -3 \\ 2 \\ -6 \end{bmatrix}$$

Column vector

- **Column vector** is defined as

$$\bar{\mathbf{u}} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

By default, all vectors will be column vectors

Row vector

- A **row vector** can be similarly defined as

$$\bar{\mathbf{v}} = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$$

Vector transpose

- The **vector transpose** of a column vector is defined as

$$\bar{\mathbf{u}} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \Rightarrow \bar{\mathbf{u}}^T = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix}$$

Vector transpose

- Similarly, the **vector transpose** of a row vector is defined as

$$\bar{\mathbf{v}} = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} \Rightarrow \bar{\mathbf{v}}^T = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Vector sum

- Consider two vectors $\bar{\mathbf{u}}, \bar{\mathbf{v}} \in \mathbb{R}^n$
- Their **sum** is given by

$$\bar{\mathbf{u}} + \bar{\mathbf{v}} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

Vector sum – example

- **Vector sum** example is below

$$\bar{\mathbf{u}} = \begin{bmatrix} 3 \\ -2 \\ 6 \end{bmatrix}, \quad \bar{\mathbf{v}} = \begin{bmatrix} 4 \\ 5 \\ -8 \end{bmatrix}$$

$$\bar{\mathbf{u}} + \bar{\mathbf{v}} = \begin{bmatrix} 3+4 \\ -2+5 \\ 6-8 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ -2 \end{bmatrix}$$

Vector scalar product

- The **scalar product** of $\bar{\mathbf{u}} \in \mathbb{R}^n$ with a real number k is

$$k\bar{\mathbf{u}} = k \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} ku_1 \\ ku_2 \\ \vdots \\ ku_n \end{bmatrix}$$

Scalar product – example

- **Vector scalar product** example is below

$$\bar{\mathbf{u}} = \begin{bmatrix} 3 \\ -2 \\ 6 \end{bmatrix}$$

$$5\bar{\mathbf{u}} = 5 \begin{bmatrix} 3 \\ -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \\ 30 \end{bmatrix}$$

Linear combination

- Consider vectors $\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \ldots, \bar{\mathbf{u}}_m \in \mathbb{R}^n$ with real numbers $k_1, k_2, \ldots, k_m$
- A **linear combination** is defined as

$$\bar{\mathbf{v}} = k_1\bar{\mathbf{u}}_1 + k_2\bar{\mathbf{u}}_2 + \cdots + k_m\bar{\mathbf{u}}_m$$

Example – Linear combination

- Example of **linear combination** is below.

$$\bar{\mathbf{u}} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \quad \bar{\mathbf{v}} = \begin{bmatrix} 5 \\ -1 \\ -2 \end{bmatrix}$$

$$3\bar{\mathbf{u}} - 2\bar{\mathbf{v}} = \begin{bmatrix} 3 - 10 \\ 12 + 2 \\ -9 + 4 \end{bmatrix} = \begin{bmatrix} -7 \\ 14 \\ -5 \end{bmatrix}$$

02

Vectors in Physics

Vectors as arrows: origin, addition and scaling



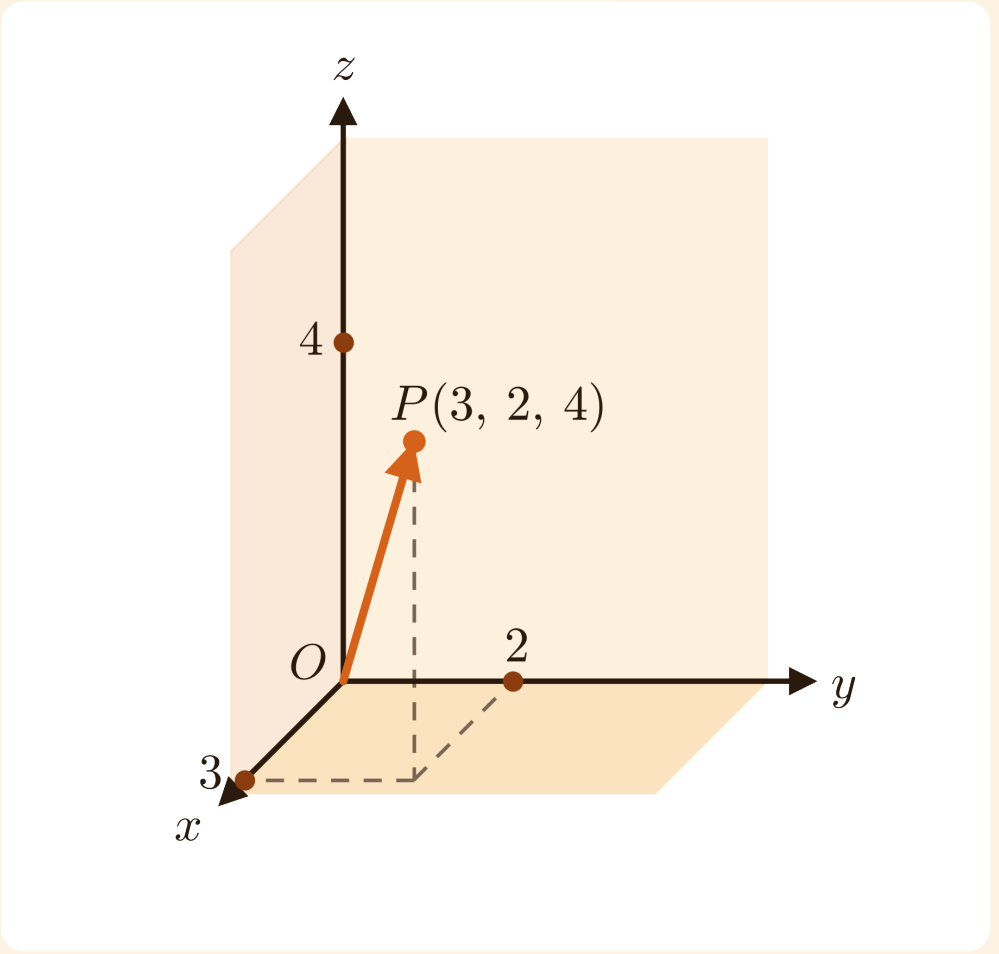
Vectors in Physics

In Physics, quantities such as **force** and **velocity** possess both **magnitude** and **direction**.

These quantities can be represented by arrows having appropriate **lengths and directions**.

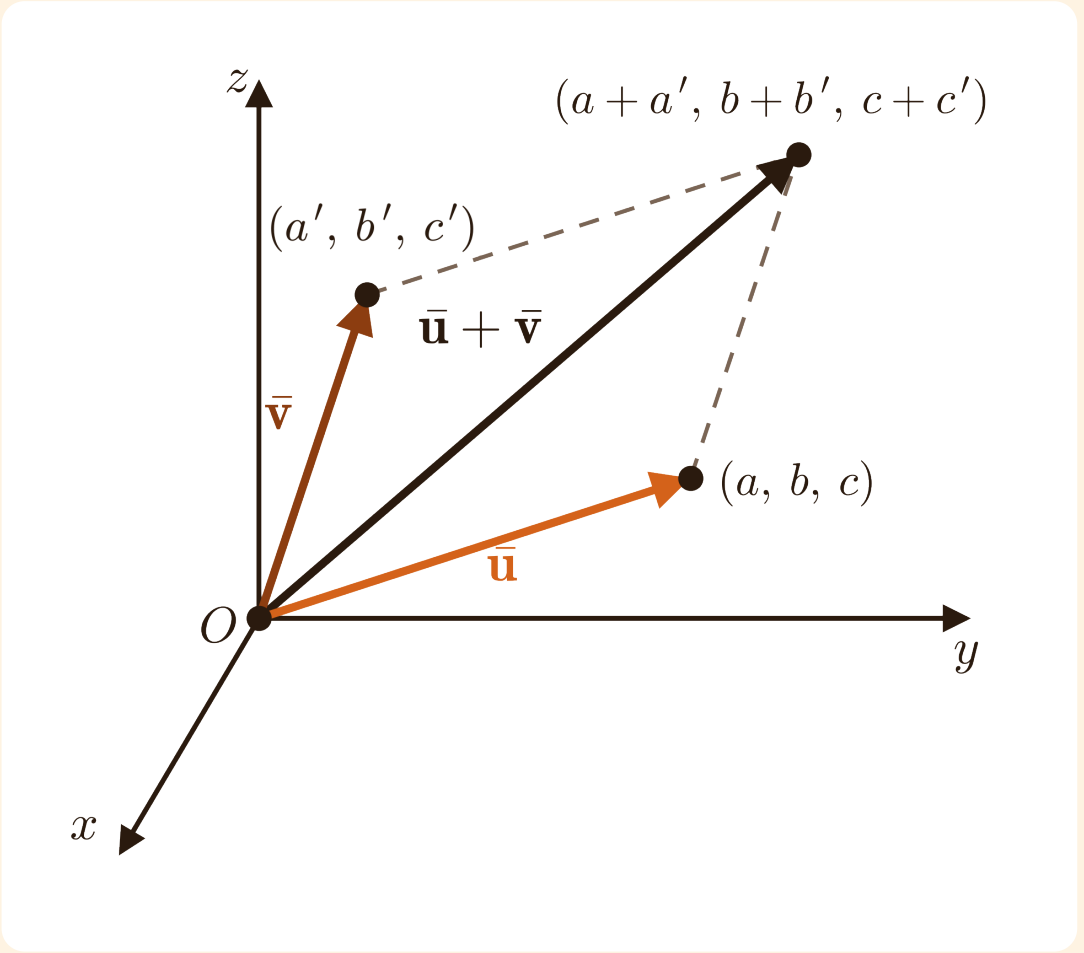
Origin

- In $\mathbb{R}^3$, the origin $[0, 0, 0]^T$ is chosen as the reference point O
- Then every vector is **uniquely determined** by the coordinates of its endpoint



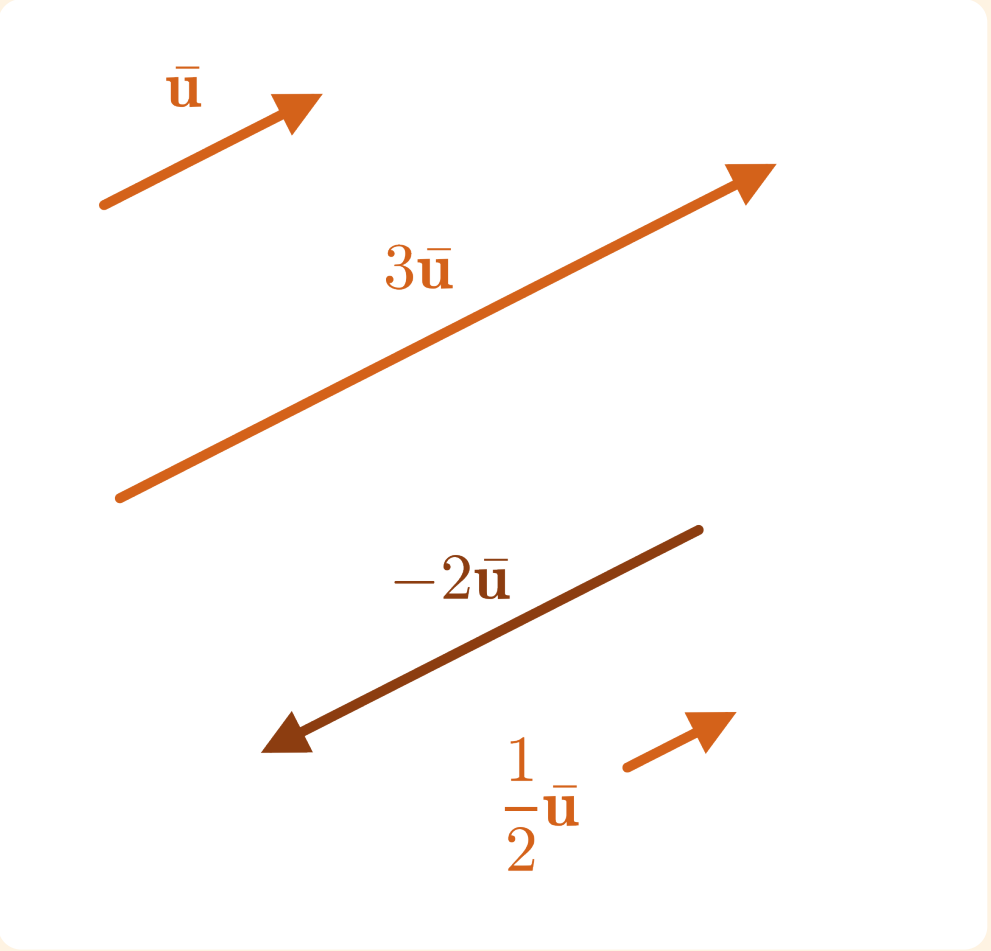
Addition of vectors

- The resultant $\bar{\mathbf{u}} + \bar{\mathbf{v}}$ of two vectors $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ is obtained by the **parallelogram law**
- That is, $\bar{\mathbf{u}} + \bar{\mathbf{v}}$ is the diagonal of the **parallelogram** formed by $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$.



Scaling of vectors

- The product $k\bar{\mathbf{u}}$ of a vector $\bar{\mathbf{u}}$ by a real number k is obtained by **multiplying the magnitude** of $\bar{\mathbf{u}}$ by k
- and retaining the **same direction** if $k > 0$ or the **opposite direction** if $k < 0$.



03

Properties of Vectors

Rules for vector addition and scalar multiplication



Basic properties

- Basic properties of vectors are below
- For any vectors $\bar{\mathbf{u}}, \bar{\mathbf{v}}, \bar{\mathbf{w}} \in \mathbb{R}^n$

$$(\bar{\mathbf{u}} + \bar{\mathbf{v}}) + \bar{\mathbf{w}} = \bar{\mathbf{u}} + (\bar{\mathbf{v}} + \bar{\mathbf{w}})$$

$$\bar{\mathbf{u}} + 0 = 0 + \bar{\mathbf{u}} = \bar{\mathbf{u}}$$

$$\bar{\mathbf{u}} + (-\bar{\mathbf{u}}) = 0$$

$$\bar{\mathbf{u}} + \bar{\mathbf{v}} = \bar{\mathbf{v}} + \bar{\mathbf{u}}$$

Basic properties

- For any vectors $\bar{\mathbf{u}}$, $\bar{\mathbf{v}}$, $\bar{\mathbf{w}} \in \mathbb{R}^n$ with real numbers k, k'

$$k\left(\bar{\mathbf{u}}+\bar{\mathbf{v}}\right)=k\bar{\mathbf{u}}+k\bar{\mathbf{v}}$$

$$\left(k+k'\right)\bar{\mathbf{u}}=k\bar{\mathbf{u}}+k'\bar{\mathbf{u}}$$

$$\left(kk'\right)\bar{\mathbf{u}}=k\left(k'\bar{\mathbf{u}}\right)$$

$$1\bar{\mathbf{u}}=\bar{\mathbf{u}}$$

04

Dot Product

Inner product, orthogonality and its properties



Dot product

- Consider vectors

$$\bar{\mathbf{u}} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad \bar{\mathbf{v}} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

- Their **dot product** or **inner product** is defined as

$$\bar{\mathbf{u}} \cdot \bar{\mathbf{v}} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Dot (inner) product

- The **dot product** can also be defined as

$$\bar{\mathbf{u}}^T \bar{\mathbf{v}} = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$= u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Dot product – example

- Example of **dot product** is below

$$\bar{\mathbf{u}} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \quad \bar{\mathbf{v}} = \begin{bmatrix} 5 \\ -1 \\ -2 \end{bmatrix}$$

$$\bar{\mathbf{u}} \cdot \bar{\mathbf{v}} = 1 \times 5 + 4 \times (-1) + (-3) \times (-2)$$

$$= 5 - 4 + 6 = 7$$

Orthogonality

- Vectors $\bar{\mathbf{u}}$, $\bar{\mathbf{v}}$ are **orthogonal** (or **perpendicular**) if their dot product is **zero**, i.e.,

$$\bar{\mathbf{u}} \cdot \bar{\mathbf{v}} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = 0$$

Orthogonality – example

- Consider vectors $\bar{\mathbf{u}}$, $\bar{\mathbf{v}}$ below and determine k such that they are orthogonal

$$\bar{\mathbf{u}} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 4 \end{bmatrix}, \quad \bar{\mathbf{v}} = \begin{bmatrix} 5 \\ k \\ -7 \\ -3 \end{bmatrix}$$

$$\bar{\mathbf{u}} \cdot \bar{\mathbf{v}} = 10 + 3k - 7 - 12 = 0$$

$$\Rightarrow 3k = 9 \Rightarrow k = 3$$

Dot product – properties

- The **dot product** has the following properties

$$(\bar{\mathbf{u}} + \bar{\mathbf{v}}) \cdot \bar{\mathbf{w}} = \bar{\mathbf{u}} \cdot \bar{\mathbf{w}} + \bar{\mathbf{v}} \cdot \bar{\mathbf{w}}$$

$$\bar{\mathbf{u}} \cdot \bar{\mathbf{v}} = \bar{\mathbf{v}} \cdot \bar{\mathbf{u}}$$

$$(k\bar{\mathbf{u}}) \cdot \bar{\mathbf{v}} = k\,(\bar{\mathbf{u}} \cdot \bar{\mathbf{v}})$$

Summary

- A vector $\bar{\mathbf{u}} \in \mathbb{R}^n$ is an n-tuple of real numbers, a column by default
- The transpose $\bar{\mathbf{u}}^T$ turns a column vector into a row vector, and vice versa
- Vectors add entry by entry; $k\bar{\mathbf{u}}$ multiplies every entry by k
- A linear combination of vectors is $k_1\bar{\mathbf{u}}_1 + k_2\bar{\mathbf{u}}_2 + \cdots + k_m\bar{\mathbf{u}}_m$
- Geometrically, $\bar{\mathbf{u}} + \bar{\mathbf{v}}$ follows the parallelogram law; $k\bar{\mathbf{u}}$ flips direction if $k < 0$
- Vector addition is commutative and associative; scalars distribute over sums
- Dot product: $\bar{\mathbf{u}} \cdot \bar{\mathbf{v}} = u_1v_1 + u_2v_2 + \cdots + u_nv_n = \bar{\mathbf{u}}^T\bar{\mathbf{v}}$; orthogonal if $\bar{\mathbf{u}} \cdot \bar{\mathbf{v}} = 0$



Thank You!

Questions?